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ELEMENTS

OF

PLANE TRIGONOMETRY,

WITH ITS APPLICATION TO

MENSURATION OF HEIGHTS AND DISTANCES,

SURVEYING AND NAVIGATION.

BY

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In the following treatise the object has been to present the subject of Plane Trigonometry in a natural and connected order, and in general to lead the learner to feel the want of a new principle before proceeding to its investigation. After the necessary definitions the learner is taught to solve a triangle by construction, the method which naturally first occurs. From the obvious imperfection of this method, he is led to see the necessity, where accuracy is desired, of a solution by arithmetical calculation. But here a difficulty presents itself, arising from the impracticability of uniting in the same calculation the incongruous elements of straight lines and angles. In the effort to surmount this difficulty he is led to see the manner in which, first the sines and cosines, and secondly the tangents and secants are introduced, and made subservient to the object before him. These are found to be simply parts in an indefinite series of right angled triangles, themselves susceptible of calculation, and by means of which other similar triangles, and finally all plane triangles may be solved. The theory of the trigonometrical tables is thus developed, and the object proposed, the complete solution of a plane triangle, accomplished.

The principles of Plane Trigonometry having thus been obtained, are next applied to the mensuration of inaccessible Heights and Distances, to Surveying and Navigation, and in a manner adapted, it is hoped, to exhibit clearly their great practical value.

In the Surveying, in particular, great care has been taken in the preparation of examples and arrangement of the subject, gradually to lead the learner from the more simple survey of an ordinary field to that of routes for canals and railroads, the survey of the Public Lands, the method of exhibiting on paper the contour and accidents of the earth's surface, the preparing of maps of towns, counties, &c.; and finally the process by which the circumference of the earth is determined.

The work will form a part of the College course of Mathematics. Constant reference, however, has been had in its preparation to the wants of pupils in our Academies and High Schools, for whose use it will be found, it is believed, sufficiently simple. WM. SMYTH.

BOWD. COLL., NOV. 1852.

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PLANE TRIGONOMETRY.

SECTION I.

PRELIMINARY REMARKS.

1. A figure bounded by three lines is called a *triangle*.

2. A triangle ABC (fig. 1), in which the bounding lines are straight lines, is called a *plane triangle*.

3. A plane triangle DEF (fig. 2), in which one of the sides DE forms with another DF a right angle, is called a *right angled triangle*. Any other triangle ABC (fig. 1), in which neither of the angles is a right angle, is called an *oblique angled triangle*.

4. In the right angled triangle DEF (fig. 2), the part DF opposite the right angle, is called the *hypothenuse;* the other two parts DE, EF are called the *sides;* sometimes also the *perpendicular and base*.

5. In a plane triangle there are six parts, viz. three angles and three sides. In order to determine the others, certain of these parts must be given.

6. A single part, it is evident, is not sufficient to determine the rest. Let us begin with two.

And first let the lines a and b (fig. 3), be two of the sides of a triangle. The lines a and b, it is evident, may make with each other any angle whatever. They may, therefore, be sides in an indefinite number of triangles, ABC, ABC,′ &c. The other parts of a triangle cannot, therefore, be determined, when two of the sides are the only parts which are given.

Again, let a side AB, and an angle A (fig. 4), be two given parts of a triangle. The other parts cannot be determined by these; for the side AB and the angle A may belong to an indefinite number of triangles ABC, ABC,′ &c.

Moreover, if two of the angles, or, which is the same thing, if all the angles are given, the remaining parts will still be undetermined; for in the triangle ABC (fig. 5) if the lines B′C′, B″C″, &c. are drawn parallel to BC, an indefinite number of triangles ABC, AB′C′, &c. may be formed, in each of which the angles will be respectively equal, while the sides are of different magnitudes.

In order then to determine the remaining parts of a triangle, *three, at least, of the parts must be given, one of which must be a side.*

7. It will be seen in what follows that the remaining parts of a triangle may always be determined, when three of the parts are given, provided that one of these is a side.

That branch of science which teaches how to find the remaining parts of a plane triangle from certain parts which are given, is called *Plane Trigonometry.*

SECTION II.

SOLUTION OF TRIANGLES BY GEOMETRICAL CONSTRUCTION.

8. The method of solving a triangle from certain of the parts which are given, which first suggests itself, is, to con-

struct the triangle by means of the given parts, and then to measure the required parts, in the use of the instruments by which the triangle is constructed.

9. Among the parts of a triangle there are two kinds of quantities, viz. sides and angles, for each of which measures must be found.

10. The sides of a plane triangle being right lines, are expressed in the usual linear measures, as feet, yards, &c. To represent these measures for the purpose of calculating a triangle geometrically, any line AB (fig. 6) may be divided into a convenient number of equal parts; one of these being considered as unity may be further divided into equal portions for fractional parts of unity. A line divided in this manner is called a *scale of equal parts.*

Thus in figure 6, the line AB is divided into eleven equal portions; of these the extreme one on the left is divided into ten equal parts, the remainder being numbered 1, 2, 3, &c. The principal divisions of this scale may be considered as feet, miles, &c.; then the subdivisions will be tenths of a foot, mile, &c. The principal divisions may also be regarded as ten feet, ten miles, &c.; in this case the subdivisions will represent feet, miles, &c.

A convenient construction for a scale of equal parts is represented figure 7. It consists, when intended for a decimal scale, of eleven lines drawn parallel to each other at equal distances, and divided into convenient portions by perpendicular lines. In the extreme divison on the left, the upper and lower lines are divided into ten equal parts, and the subdividing points are connected by *diagonal* lines, that is, by lines drawn from the first point on the lower line to the second on the upper. Then by similar triangles $ac : ab :: cd : bo$, that is, cd is one tenth of the subdivisions, or one hundreth of the primary divisions of the scale. In like manner, the corresponding portion on the second parallel line may be shown to be two hundreths, and that on the third parallel line three hundreths of the primary divisions of the scale.

Let it be required, for example, to take off the number 255 from this scale. Considering the subdivisions as containing ten each, we extend the dividers from figure 2 of the principal divisions to 5 of the subdivisions for 250; then by opening the dividers to the corresponding extent on the fifth of the parallel lines, we shall obtain the length required.

11. In order to estimate the magnitude of the angles, it would seem more natural to refer them to the right angle, as the unit of measure; since, in this case, the quantities to be measured would be the same in kind with the quantity assumed as their measure. It is found, however, more convenient in practice to measure angles by arcs of circles. In the circle AC (fig. 8), in whatever ratio the angle DBC at the centre increases or diminishes, the arc CD, on which it stands, increases or diminishes in the same ratio (Geom. B. 3); the arc CD may therefore be assumed, as the measure of this angle. In like manner any other concentric arc EF, intercepted by the sides of the angle DBC, may be used as its measure. The arc CD, indeed, considered as a magnitude of a certain length, is manifestly greater than the arc EF. In the measure of angles, however, it is not the absolute value of the arcs, which we regard, but merely their ratio to an entire circumference. Since then the arc EF is to the circumference EG, as the arc CD to the circumference CA; considered as parts of the entire circumferences, to which they respectively belong, the arcs CD and EF are equal in quanity. *The measure of an angle is then the arc of a circle, having its centre in the vertex of the angle, and intercepted by the lines, which form its sides.* In the measure

of an angle it is therefore immaterial, what may be the size of the circle employed, provided it cut the sides of the angle. In comparing different angles, however, the arcs, on which they are measured, should be described with equal radii.

12. The entire circumference of every circle, whether great or small, is divided into 360 equal parts called degrees, each degree into 60 equal parts called minutes, each minute into 60 equal parts called seconds, each second into 60 equal parts called thirds, &c. The character used to denote degrees is $^\circ$, placed at the right hand of the unit figure and a little above it, that for minutes is $'$, that for seconds is $''$, that for thirds is $'''$, so that $40^\circ\,30'\,25''\,10'''$ signifies 40 degrees, 30 minutes, 25 seconds, and ten thirds. This division, however, is entirely arbitrary, and any other may be assumed at pleasure. We may, for instance, after the manner of the French, divide the circle into 400 equal parts for degrees, and each degree into 100 equal parts for minutes, with further subdivisions according to the principles of the decimal progression.

13. We have seen art. 11, that the measure of any angle DBC (fig. 8) is the arc CD, of a circle CA; but the chord CD, being the distance in a right line between the extremities of the arc CD, will determine the magnitude of this arc. If then the several chords for each degree, minute, &c., contained in the arc AB (fig. 9), equal to a quadrant or fourth part of a circle, be transferred to the line BD, the line BD, denominated in this case a *line of chords*, may with convenience be employed in the measurement of angles.

Let it be required for example, to ascertain the magnitude of the angle ABC (fig. 10) by the line of chords. With the extent from 0 to 60 on the line of chords, we describe from B as a centre, an arc cutting the two lines, which include the angle in the points m and n; then since the chord of 60° is equal to radius (Geom. B. 5, P. 4) this arc will have the same radius with that, to which the given scale of chords belongs; consequently, the distance between the points of intersection *m* and *n*, applied to the line of chords, will give the measure of the angle in degrees.

Conversely, let it be required to make an angle ABC of a given magnitude, 30° for example. Having drawn a line BC, from the point B with a radius B*n*, equal to the chord of 60° on the line of chords, describe an arc *nm*; the extent from B to 30° on the line of chords applied from *n* on the arc *nm* will give the point, through which if the line BA be drawn, the angle ABC will be formed of the required magnitude.

If the angle required is greater than 90°, with the chord of 90, we first set off on *mn* an arc of 90°; then from this setting off the excess of the given angle above 90°, we shall have the point through which to draw the remaining side of the angle.

14. We are now prepared for the solution of triangles, when three, the required number of parts, are given. The different Problems which may occur are 1°. Given the three sides, 2°, two sides and an angle, 3°, one side and two angles, to find the remaining parts.

PROBLEM I.

1. Given *three sides* of a triangle, to find the remaining parts.

Ex. 1. Let the sides of the triangle be 39, 45, and 37 feet respectively.

By means of the scale of equal parts we lay down one of the sides AB, for example (fig. 1), equal to 45. From the same scale we then take in the dividers one of the remaining sides 39, and with one foot in the point A, as a centre, we describe an arc. Taking next in the dividers the remaining side 37, from the point B, as a centre, we describe another arc cutting the former in the point C. We then draw AC and BC, and the triangle ABC will be the triangle required. The angles may then be measured upon the line of chords in the manner explained above.

Ex. 2. Given the three sides 53.5, 82.08, and 46.7; to find the angles.

Ex. 3. Given the three sides 67, 75.9, and 121.05; to find the angles.

PROBLEM II.

2. Given *two sides and an angle*, to find the remaining parts.

The problem admits of several cases.

1. Two sides and the included angle being given.

Ex. 1. Two sides of a triangle are 60, and 100 rods respectively, and the angle included is 45° 30′. What are the remaining parts?

We draw first one of the given sides BC, for example, (fig. 11) equal to 60. At the point B, one of the extremities of BC, we make next an angle ABC equal to 45° 30′, the given angle; through this we then draw the other given side BA equal to 100; and connecting the points A and C, ABC will be the triangle required. The required parts may then be measured, the side AC on the scale of equal parts, and the angles A and C upon the line of cords.

Ex. 2. Given two of the sides of a triangle equal to 354.7, and 563.03 feet respectively, and the angle contained between them 53° 45′, to find the remaining parts.

Ex. 3. Two sides of a triangle are 75.3, and 89.75 yards respectively, and the included angle is 97°. What are the remaining parts?

2. Two sides and an angle opposite the longer given side.

Ex. 1. Let the given sides be 57.5, and 103 feet respectively, and the angle opposite 103, the longer side, be 40°, to find the remaining parts.

We draw the shorter given side AB (fig. 12) equal to 57.5, and at one extremity A of this side we make the angle BAC equal to 40°, through which we draw an indefinite line AC; then with an extent equal to 103, the other given side, we describe from B, as a centre, an arc cutting the indefinite line AC at C, and join BC. ABC will be the triangle required, the remaining parts of which may be measured as above.

Ex. 2. Given one side of a triangle equal to 93 feet, another side equal to 179 feet, and the angle opposite this last equal to 63° 30′, to find the remaining parts.

3. Two sides and an angle opposite the shorter given side.

Ex. 1. Given in the triangle ABC (fig. 13) the side AB 72 feet, the side AC 41.5 feet, and the angle B 30°; required the remaining parts.

In this example, in which the side opposite the given angle is less, than the other given side the solution is *ambiguous;* for if from the point A, with a radius equal to AC, we describe the arc CC′ cutting the line BC in C′, and draw AC′, a second triangle ABC′ will be formed, which will also contain the three given parts. To determine then, which of these triangles is intended in the question proposed, we must know, in addition to the given parts, whether the angle C be acute or obtuse. With respect to this case, we may remark in passing, that the angles AC′B, ACB are together equal to two right angles. Indeed the angles AC′B, AC′C are together equal to two right angles; but by construction the triangle AC′C is isosceles; the angle ACB is therefore equal to AC′C, whence AC′B, ACB are together equal to two right angles. The angles ACB AC′B are therefore, the supplements of each other, since we understand by the *supplement of an angle that which must be added to this angle in order to make two right angles.*

Ex. 2. Given two sides of a triangle equal to 530, and 253 respectively, and the angle opposite the shorter side equal to 28°, to find the remaining parts.

PROBLEM III.

Given *one side and two angles,* to find the remaining parts.

Ex. 1. Given one side of a triangle equal to 120 yards, and the adjacent angles equal to 50°, and 63° respectively to find the remaining parts.

The solution of this problem is left as an exercise for the learner. It will be obesrved, however, that if the given angles are not both adjacent to the given side, we must subtract their sum from 180°, which will give the other adjacent angle.

Ex. 2. Given the side AB of a triangle (fig. 14), 40 feet, the angle ABC 52°, and the angle BAC 70°; to find the remaining sides.

Ex. 3. Given one side of a triangle 600 yards, the angle opposite 65°, one of the adjacent angles 40° 30′; required the remaining parts.

15. It is now evident, that we may always construct a triangle, when three of its parts are given, provided that one of them be a side. If, however, the given parts are two of the sides and an angle opposite one of them, and the side opposite the given angle is less than the other given side, to determine the triangle, we must know whether the angle sought be acute or obtuse.

16. From the preceding examples it is easy to see, that the solution, obtained by geometrical construction, will not, on account of the imperfection of instruments, be precisely the truth, but only a rough approximation to it. It is then required to ascertain methods of calculating triangles by numerical operations, since by means of these the calculation may be carried to any degree of exactness at pleasure.

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SECTION III.

OF THE SOLUTION OF TRIANGLES BY ARITHMETICAL CALCULATION.

17. In the solution of a triangle, it is obvious, that the magnitude of the angles will always come into consideration. Thus in the triangle ABC (fig. 1), if it be required to determine the third side AC, the angle at B and the sides AB, BC being given, since the value of AC evidently depends in part upon the magnitude of the angle at B, it is manifest, that the magnitude of this angle must be made to enter into the computation. But the angle at B is measured on the arc of a circle, while the sides are right lines; of consequence the parts, which are necessarily the elements of the calculation, are different in kind and do not admit of comparison with each other. In the effort then, by means of numerical operations, to determine the different parts of a triangle, we are embarrassed at the outset by the difficulty of introducing into the calculation the magnitude of the angles.

18. To show, in a particular case, the manner in which this difficulty is removed, we suppose that the triangles ABC, *abc* (fig. 15) are similar, and that the sides *ab*, *bc*, *ac*, of the latter are known; if then in the triangle ABC a side BC be given, it is evident from the nature of similar triangles,

that the remaining parts of this last may be determined from those of the other by means of a simple proportion.

Let it then be supposed, for further illustration, that we have determined, by geometrical construction, the ten triangles in Plate II, the first having the angle at B 10°, the second 15°, the third 20°, and so on; and, for the sake of simplicity, let these triangles each be right angled at C. If it now be proposed to solve the triangle *abc* (fig. 16), right angled at *c*, the side *bc* 73 feet, and the angle *b* 40°, it is evident, on inspecting the series of computed triangles, that the triangle No. 7 is similar to the triangle proposed; since these triangles are both right angled, and one of the acute angles in the one is equal to an acute angle in the other. By means then of the computed triangle in the series, the remaining sides in the proposed triangle may be determined.

Thus to find *ac*, we have the proportion, as the side BC is to AC in the computed triangle, so is the side *bc* to *ac* in the triangle proposed, that is, as 90 : 72 : : 73 : 58.4 the side *ac* required; and to find *ab* we have BC : AB : : *bc* : *ab*, or 90 : 115 : : 73 : 93.3 the side *ab*. In like manner we may determine, any right angled triangle whatever, provided that one of its acute angles be equal to an acute angle in one of the triangles of the series. We may form, moreover, a more extended series of right angled triangles, by making in the first triangle an angle of one degree, in the second an angle of two degrees, in the third of three, and so on, to eighty-nine degrees inclusive; in which case, there will always be found, among the triangles of this series, one similar to any right angled triangle, which may be proposed; provided that one of its acute angles be an entire number of degrees.

In like manner, it is evident, we may so extend the series of computed triangles, that there will always be found, among the number, a triangle similar to the one, which shall be proposed for calculation; of consequence, as in the preceding example, the parts of this last may be determined from the former by means of a simple proportion.

19. Thus far we have supposed the series of triangles actually constructed, and arranged as in Plate II, for the calculation of other similar triangles; such an arrangement, however, would obviously be inconvenient, especially as the series is more extended. To avoid this inconvenience, we may write in a table the different parts in the several triangles of the series, expressing those, which correspond in each by the same general term, and arranging them in a uniform order.

20. To form a table, for example, of the triangles, which we have calculated Plate II, we write down in order in the first column, as below, the angles at B in the several triangles, in the second the hypothenuses, in the third the bases, and in the fourth the perpendiculars, recollecting that by the term base we understand the side adjacent the angle at B in each of the triangles, and by the term perpendicular the side opposite this angle.

Deg.	Hyp.	Base	Perp.
10	242	237	43
15	127	123	33
20	121	113	42
25	139	125	59
30	132	114	65
35	125	104	70
40	115	90	72
45	103	72	72
50	93	61	69
55	132	73	109

21. To apply the table, let it be supposed, that we know in the right angled triangle ABC (fig 17) the angle at B 35°, and the side BC 200 feet; in order to solve this triangle, we look along the column of degrees for 35; against

this we find 104 the side of the tabular triangle corresponding to the given side in the triangle proposed ; then, as 104 the base in the tabular triangle : 125 the hypothenuse of the same triangle : : BC 200 feet : 240.3 feet, the hypothenuse of the triangle proposed. And as base 104 : perp. 70 : : 200 : 134.6, the side AC of the proposed triangle.

Again, let there be given the hypothenuse 186 and the base 12~~1~~2, to find the angle. These numbers have, it will be seen, the same proportion as 93 to 61. Against these last in the table we find 50°. The angle sought is, therefore, 50°.

SECTION IV.

METHOD OF FORMING A SERIES OF TRIANGLES, HAVING ANGLES OF EVERY POSSIBLE MAGNITUDE.

22. Having seen the use, which may be made of a series of triangles containing angles of every possible magnitude, and the sides of which are calculated ; we next inquire into the method of constructing such a series. For the sake of simplicity let the triangles to be determined be right angled. It is evident (fig. 18), that such a series of triangles may be constructed in the quadrant of a circle. For if from each point in the arc AB, we let fall the perpendiculars ED, E′D′ E″D″, &c. upon the radius AC, and draw the radii EC, E′C, E″C, &c., the triangles EDC, E′D′C, E″D″C, &c. thus formed are right angled at D, D′, D″ &c., and the angles ECD, E′CD′, E″CD″ &c. will be successively of every possible magnitude. And since any two right angled triangles are similar, when an acute angle in one is equal to one of the acute angles in the other, there cannot be proposed a right angled triangle, which will not be similar to some one of the triangles constructed in this manner.

23. To form a table from the triangles of this series, we employ, as before, convenient terms to express their sides. The hypothenuses CE, CE′, CE″, &c. being all radii of the same circle, may be designated by the term *radius.* The perpendiculars ED, E′D′, E″D″, &c. manifestly increase with the angles ECD, E′CD′, E″CD″, &c., to which they are opposite, or which is the same thing, with the arcs AE, AE′, AE″ &c., by which these angles are measured. On account of this connection we designate the perpendiculars ED, E′D′, E″D″, &c., by the term *sine.* We then define the sine of an arc, *the perpendicular let fall from one extremity of this arc upon the radius, which passes through the other extremity.*

In order to decide upon a term to express the remaining side, we now remark, that two *arcs are said to be complements the one of the other, when the sum, or difference of these arcs, is equal to the fourth part of the circumference of a circle.* From the definition already given of a sine, the line EF, considered with reference to the arc EB, is the sine of that arc; but EB is the complement of AE; EF therefore is the sine of the complement of AE, and may for the sake of brevity be called its *cosine.* But the side CD of the triangle CDE is equal to EF, since these lines are parallel to each other, and are included between parallels; CD may therefore be considered the cosine of AE, and CD′ of AE′ &c.; whence the remaining side in the triangles CDE, CD′E′, &c. may be designated by the term *cosine*; and we say, that *the cosine of an arc is the sine of the complement of this arc, and is equal to that part of the radius comprehended between the centre and the foot of the sine.*

24. From what has been said, it will now be recollected, that in constructing, in the quadrant of a circle, a series of right angled triangles having angles of every possible magnitude, the radius of the quadrant forms the common hypoth-

enuse in the several triangles, and that the remaining sides are respectively the sines and cosines of the acute angle, which has its vertex at the centre.

……

SECTION V.

METHOD OF CALCULATING THE SIDES IN THE SERIES OF TRIANGLES, CONSTRUCTED IN THE QUADRANT OF A CIRCLE.

25. We have now ascertained a convenient method of constructing a series of triangles, in which the angles are of every possible magnitude. But the triangles of this series being determined by construction, no higher degree of accuracy is attained in the use of them. Let us next see if rules can be found by which they can be determined by arithmetical calculation.

And first, it is evident, that it will be sufficient merely to know the value of the sides as compared with each other. We may, therefore, adopt the common hypothenuse, or radius, as the unit; and the question will be, to determine the values of the sines and cosines in decimal parts of this unit; or, which is the same thing, we may consider the radius as divided into 100 000, or any other convenient number of equal parts, and then determine the number of these parts in each of the sines and cosines.

26. It is evident from inspection (fig. 18), that the sine of 90° is equal to radius. And since (fig. 19) the side of an inscribed hexagon is equal to radius (Geom. B. 5); if through the centre E of the arc BC we draw the radius EF, the chord BC will be bisected at G, and BG the sine of the arc BE will be equal to half of radius; but the arc BE is equal to one third of a quadrant, or thirty degrees; there are then two

of the sines in a quadrant, the values of which we may know at the outset, viz. the sine of 90° equal to radius, and the sine of thirty degrees equal to one half of radius.

27. Beginning with the sine BG (fig. 19), we find immediately its cosine GF; for in the triangle BGF, which contains these sides, and the hypothenuse of which is equal to radius or unity, we have $BF^2 = BG^2 + GF^2$; whence $GF = \sqrt{BF^2 - BG^2}$, that is, $\cos EB = \sqrt{R^2 - \sin EB^2}$. To find then the cosine of an arc, when the sine is given; *from the square of the radius subtract the square of the sine, the square root of the remainder will be the cosine.*

28. From what has been said, the relative value of the sides in one of the triangles of the series which we wish to calculate, may now be considered as known; we are next led to inquire into the use, which may be made of these in the calculation of the remaining triangles of the series.

29. If the arc AE (fig. 20) be divided into two equal parts by the radius CD, the chord AGE will also be divided into two equal parts, and EG will be the sine of the arc DE one half of AE. The triangle AEF, right angled at F, gives $AE = \sqrt{AF^2 + EF^2}$; but $AF = AC - FC = R - \cos AE$, and $EF = \sin AE$; squaring these values of AF and EF and substituting the squares in the expression for AE we have $AE = \sqrt{\sin AE^2 + R^2 - 2R \times \cos AE + \cos AE^2}$; but $EF^2 + FC^2 = CE^2$, or $\sin AE^2 + \cos AE^2 = R^2$, hence $AE = \sqrt{2R^2 - 2R \times \cos AE}$; whence $\frac{1}{2}$ the chord AE, or $EG = \frac{1}{2}\sqrt{2R^2 - 2R \times \cos AE}$. But EG is the sine of one half the arc AE; whence to find the sine of half any arc less than a quadrant; *from twice the square of radius subtract twice the radius multiplied by the cosine of the given arc; one half of the square root of the remainder will be the sine of half this arc.*

30. Let it now be supposed that we wish to calculate a series of triangles, in which there will always be found a triangle having for one of its acute angles any possible number of minutes, or in other words, that we wish to calculate a series of triangles for every minute in the quadrant. By art. 25 we have radius equal to 1.00000, or, omitting the separatrix, to 100000; the sine of 30° will then be equal (art. 26) to 50000, and its cosine (art. 27) to 86603. By the rule last obtained we may now calculate the sine of 15°, one half of 30°, the sine of 7° 30′, one half of 15°, &c. Proceeding in this manner we obtain at the eleventh division, as below, the sine of 52″ 44‴ 3⁗ 45′′′′′ equal to 00025, or calling the radius 1000000, to 000255.

	Sine	*Cos.*
30°	50000	86603
15°	———	———
7° 30′	———	———
3° 45′	06540	99786
———	———	———
———	———	———
———	———	———
0° 14′ 3″ 45‴	00409	99999
———	———	———
———	———	———
———	———	———
0° 0′ 52″ 44‴ 3⁗ 45′′′′′	00025	———

We next inquire into the manner, in which the sine of one minute may be obtained from this last.

31. It is evident (fig. 21), that if the arcs A*b*, A*c* are very small, they will not differ materially from right lines; the triangles A *b d*, A *c e* may then be considered similar, and A*b* is to A*c*, as *bd* to *ce*. *In very small arcs therefore the ratio of the arcs is so nearly equal to the ratio of their sines, that the one may be taken for the other without sensible error.*†

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† The error in supposing that arcs less than one minute are proportional to their sines, will not affect the first ten places of decimals.

To find the sine of one minute from the sine of 52″ 44‴ 3⁗ 45′′′′′, we have then the following proportion, 52″ 44‴ 3⁗ 45′′′′′ : 60″ : : 000255 : 00029 the sine of one minute.

32. The sine and cosine of one minute may now be considered as known, and of consequence all the parts in the first triangle of the series, which we wish to calculate. We proceed to the investigation of principles, by which the remaining triangles of the series may be computed from this first.

Let it be supposed, that the sines of the two arcs AB, AC (fig. 22) are known; it is proposed to find the sines of the sum and difference of these arcs. To construct the figure, make the arc AD equal to the arc AC; draw the chord CD and the radius LA; the radius LA will divide this chord into two equal parts at the point I (Geom. B. 3); from the points C, A, I and D, let fall upon BL the perpendiculars CK, AG, IH and DF; lastly, at the points I and D, draw IM and DN, parallel to BL. Then CK, the sine of BC, the sum of the two arcs AB, AC is equal to KM + MC; but KM is equal to HI; CK is therefore equal to HI + MC, and the sum of the values of HI and MC will be equal to the sine of the sum of the two arcs AB, AC. To find the value of HI, we have LA : LI : : AG : HI, or putting for these lines what they severally denote R : cos AC : : sin AB : HI,

hence $$\mathrm{HI} = \frac{\sin \mathrm{AB} \times \cos \mathrm{AC}}{\mathrm{R}}$$

and since the triangles LAG, CIM are similar, to find MC we have LA : LG : : CI : MC, or, R : cos AB : : sin AC : MC,

and $$\mathrm{MC} = \frac{\sin \mathrm{AC} \times \cos \mathrm{AB}}{\mathrm{R}}$$

Adding together these two expressions, we have

$$\sin(\mathrm{AB} + \mathrm{AC}) = \frac{\sin \mathrm{AB} \times \cos \mathrm{AC} + \sin \mathrm{AC} \times \cos \mathrm{AB}}{\mathrm{R}}$$

Then since CD is divided into two equal parts at the point I, CN is also divided into two equal parts at the point M, and DF the sine of BD, the difference of the two arcs AB, AC is equal to KN, that is to KM — MC, or which is the same thing to HI — MC. To find then the sine of the difference of the two arcs AB, AC we substract the one from the other the values of the lines HI, MC already obtained, and

$$\sin(AB - AC) = \frac{\sin AB \times \cos AC - \sin AC \times \cos AB}{R}$$

Having calculated therefore the cosines of any two arcs, the sines of which are given, in order to find the sine of the sum and of the difference of these arcs; *we multiply the sine of the first by the cosine of the second, and the sine of the second by the cosine of the first; the sum of the two products divided by radius will be the sine of the sum of the two arcs; and the difference of the same products divided by radius will be the sine of the difference of these arcs.*

33. The sine of the sum and of the difference of any two arcs being calculated by the rule now obtained, the cosines may be calculated by art. 27. When, however, the sines of any two arcs AB, AC (fig. 22) are given, the cosine of the sum, and of the difference of these arcs may be calculated in a more convenient manner.

Since DC (fig. 22) is divided into two equal parts in I, FK will be divided into two equal parts in H, and LK, the cosine of BC, the sum of the arcs AB, AC, will be equal to LH — HK, or to LH — IM. To find LH, we have LA : LI : : LG : LH, that is, R : cos AC : : cos AB : LH, and

$$LH = \frac{\cos AB \times \cos AC}{R}$$

To find IM, the similar triangles LAG, CIM give LA : AG : : CI : IM, that is, R ; sin AB : : sin AC : IM, and

$$IM = \frac{\sin AB \times \sin AC}{R}$$

Subtracting this last expression from the former, we have

$$\cos(AB + AC) = \frac{\cos AB \times \cos AC - \sin AB \times \sin AC}{R}$$

Moreover LF, the cosine of BD, the difference of the arcs AB, AC, is equal to LH + HF, or since HF is equal to IM, to LH + IM. To find therefore the cosine of the difference of the arcs AB, AC, we add together the values of LH and IM already found, and

$$\cos(AB - AC) = \frac{\cos AB \times \cos AC + \sin AB \times \sin AC}{R}$$

To find therefore the cosine of the sum, and of the difference of any two arcs, whose sines are given, having calculated the cosines of these arcs, *we take the product of the cosines, and also of the sines; the difference of these two products divided by radius will be the cosine of the sum of the two arcs, and the sum of the same products divided by radius will be the cosine of their difference.*

34. From the equation (art. 32)

$$\sin(AB + AC) = \frac{\sin AB \times \cos AC + \sin AC \times \cos AB}{R}$$

we may find the sine of twice any arc, whose sine and cosine are given; for, as the arcs AB, AC may be of any dimensions, they may be considered equal; substituting then AB for its equal AC throughout, we have

$$\sin 2\,AB = \frac{2 \sin AB \times \cos AB}{R}$$

To find then the sine of twice any arc, whose sine and cosine are given; *we multiply twice the sine of this arc by its cosine, and divide the product by radius.*

35. From the equation (art. 33),

$$\cos(AB + AC) = \frac{\cos AB \times \cos AC - \sin AB \times \sin AC}{R}$$

we may likewise find the cosine of twice any arc, when its sine and cosine are known; for supposing the two arcs equal, as before, and substituting AB for its equal AC throughout, we have

$$\cos 2\,AB = \frac{\cos AB^2 - \sin AB^2.}{R}$$

Hence, when the sine and cosine of an arc are known, to find the cosine of twice that arc; *we substract the square of the sine of the arc from the square of its cosine, and divide the remainder by radius.*

36. We have seen (art. 31) the manner in which the sine of one minute may be calculated, consequently, all the parts in the first triangle of the series, which we wish to compute, may be considered as known. By the rules (art. 34, and 35) for the sine and cosine of twice any arc, we may now obtain from this first triangle the sine and cosine of two minutes, or the second triangle, and by the repeated application of the rules art. 32, and 33, the remaining triangles of the series may be calculated.

Ex. From the data given below it is required to calculate, the sine and cosine of 10°, 11°,&c., extending the calculation to six places of figures, that is, supposing the radius to be divided into 1000000 equal parts.

	Sin.	*Cos.*
1°	017452	999848
5°	087156	996195
10°	———	———
11°	———	———
12°	———	———
13°	224951	974370
14°	———	———
15°	———	———

37. We now perceive the advantage of constructing the series of triangles, which we wish to compute, in the quad-

rant of a circle ; since, if the radius of the quadrant be considered as unity, the value of one of the sides in a triangle of the series will be known, and from this, radius being common to all the triangles, the remaining sides in each of the triangles of the series may be determined.

38. Having calculated a series of triangles by the rules, which we have now obtained, the different parts of this series may be arranged in a table as in art. 20. The radius, however, which is common to all the triangles may be omitted, or written once only at the head of the table.

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SECTION VI.

SOLUTION OF TRIANGLES BY THE TABLE OF SINES AND COSINES.

39. To apply the table of sines and cosines, let there be given for solution the right angled triangle ABC (fig. 23), in which the angle at B is 30°, and the hypothenuse AB three yards.

Since we have considered the radius of the table as unity without reference to any particular denomination of measure, it may manifestly, in all cases, be regarded as a unit of that species, in which the sides of a proposed triangle are expressed. With an extent then equal to one yard, which in the present instance will be the radius of the table, draw from B the arc DF ; from D let fall DE perpendicular to BC ; then will DE be the sine, and BE the cosine of the angle at B, and DBE will be the triangle of the table similar to the proposed triangle ABC. To find the side AC, we then have BD : DE : : BA : AC, that is, R : sin B : : BA : AC ; and to find BC, BD : BE : : BA : BC, that is, R : cos B : : BA : BC. Since the angle at A is the complement of the angle at B, the cosine of B is the same with the sine of A ; these two proportions may therefore be united into one, and enunciated as follows ; *radius is to the sine of one of the acute angles in a right angled triangle, as the hypothenuse to the side opposite this angle.*

40. Let it be proposed to solve the right angled triangle CDE (fig. 24).

1. Given the hypothenuse CD 250 feet, and the angle at D 30° ; required the remaining parts. To find CE, we have

$$R : \sin D :: CD : CE$$

$$\text{or } 100000 : 50000 :: 250 \text{ ft.} : 125 \text{ ft.}$$

And to find DE,

$$R : \sin C :: CD : DE$$

$$\text{or } 100000 : 86603 :: 250 \text{ ft.} : 216.5075 \text{ ft.}$$

2. Given the hypothenuse CD 250 feet, and the side CE 125 feet ; required the remaining parts.

3. Given the side DE 216.5075 feet, and the angle D 30° ; to find the remaining parts.

41. From the preceding example, it is evident, that we may determine, by means of a table of sines and cosines, one of these three things, when two of them are given, viz. *the hypothenuse, a side and one of the acute angles.*

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SECTION VII.

SECOND METHOD OF CONSTRUCTING AND CALCULATING A SERIES OF TRIANGLES, HAVING ANGLES OF EVERY POSSIBLE MAGNITUDE.

42. Let there be given the two sides DE, EC in the triangle CDE (fig. 24), to find the hypothenuse, and one of the acute angles. It will soon be perceived, that the case now proposed cannot be solved by the table of sines and cosines. This table, indeed, may contain a triangle similar to the tri

angle proposed ; from the data now given, however, no part of that triangle can be known, except the radius or hypothenuse; but neither of the sides DE, EC can be compared with radius ; of consequence from the parts now given the remainder cannot be determined.

43. For the solution of this case, it is then evident, that a series of triangles must be so constructed, that the radius shall be a side. In order to construct such a series, from the extremity of the radius AC (fig. 18) we raise the indefinite tangent AH, and draw from the centre C through each point in the quadrant AB the secants CG, CG′, CG″, &c. It is evident, that the triangles CAG, CAG′, CAG″, &c. right angled at A, must have all the combinations of angles, which can exist in a right angled triangle. The side AC moreover, common to all the triangles, will be the radius of the quadrant.

44. In the application of terms to express the different sides in the triangles of such a series, the common side AC, for the same reason as in the former case, may be called *radius.* The portions AG, AG′, AG″, &c. of the indefinite tangent AH, or in other words the perpendiculars in the triangles CAG, CAG′, CAG″, &c. are tangents of the arcs AE, AE′, AE″, &c., since we define the tangent of an arc, *a perpendicular drawn to the radius at one extremity of the arc, and terminated by the radius produced which passes through the other extremity.* The term *tangent* may therefore be applied to designate the perpendiculars of the triangles. The hypothenuses of the same triangles are secants of the arcs AE, AE′, AE″, &c., since we define the secant of an arc, *the radius, drawn through one extremity of this arc and produced, until it meets the tangent, drawn through the other extremity.* The hypothenuses of the triangles may in consequence be called *secants.*

45. We proceed to develop rules for the calculation of

the sides in this second series of triangles. The triangles CDE, CAG (fig. 18), having the angle C common, are similar; the sides of the latter may therefore be deduced from those of the former.

To find AG, we have CD : DE : : CA : AG, that is, cos AE : sin AE : : R : tang AE, hence

$$\text{tang AE} = \frac{\text{R} \times \sin \text{AE}}{\cos \text{AE}}$$

In order therefore to find the tangent of an arc, when the sine is given; *we multiply the radius by the sine of the arc, and divide the product by its cosine.*

To find CG, we have CD : CE : : CA : CG, that is, cos AE : R : : R : sec AE, hence

$$\sec \text{AE} = \frac{\text{R}^2}{\cos \text{AE}}$$

To find therefore the secant of an arc, the cosine being given; *we divide the square of the radius by the cosine of the arc.*

46. From what has been said, it is evident, that having calculated a series of triangles in the quadrant of a circle, in which the radius is the hypothenuse of the triangles, we may by the rules now obtained calculate a second series of triangles similar to the former, in which the radius shall be a side. The parts in this second series may, moreover, be arranged in a table, as before, or which is the same thing, the table, already formed from the first series of triangles, may be extended by writing down against the several angles in this table the tangents and secants of these angles.

SECTION VIII.

SOLUTION OF TRIANGLES BY THE TABLE OF TANGENTS AND SECANTS.

47. To show the use, which may be made of the table of tangents and secants, from the point F (fig. 23) draw the perpendicular FG; then will FG be the tangent, and BG the secant of the angle B; BF will be the radius, and GBF the tabular triangle similar to ABC.

To find AC, we have BF : FG : : BC : AC, that is, R : tang B : : BC : AC; a proportion, which we may thus enunciate; *radius is to the tangent of one of the acute angles in a right angled triangle, as the side of the right angle adjacent to this acute angle is to the side opposite.*

48. Ex. 1. In the triangle ABC (fig. 25) given the side BC 500 yds, and the angle at B 22° 30′; required the side AC.

To find AC, R : tang B : : BC . AC
or 100000 : 41421 : : 500 : 207.105

2. Given AC 207.105, and BC 500 yds, to find the angle at B.

By means of the tangents then, we may determine one of these three things, when two of them are known, viz. *the two sides of a right angle and an acute angle.*

49. It remains to show the use, which may be made of the secants. In the triangle ABC (fig. 23) we have BF : BG : : BC : BA, that is, R : sec B . : BC : BA, a proportion, which we thus enunciate; *radius is to the secant of one of the acute angles in a right angled triangle, as the side of the right angle adjacent to this acute angle is to the hypothenuse.*

Ex, 1. In the triangle ABC (fig. 25) given BC 500 yds, and the angle B 22° 30′, to find AB,

R : sec B : : BC : AB
100000 : 1.08239 : : 500 : 541.195

2. Given AB 541.195 yds, and the angle B 22° 30′, to find BC.

3. Given AB 541.195, and BC 500 yds., to find the angle B.

By means of the secants the same things are determined, as by the sines and cosines; the secants are in consequence but seldom employed in the solution of triangles.

50. The preceding examples, it will be perceived, do not comprehend the case, in which any two of the sides are given to find the third. This case, however, may be solved by means of the known property of a right angled triangle, viz. the square of the hypothenuse is equal to the sum of the squares of the two sides. It may moreover be solved with facility by means of the two propositions art. 39, and 47.

In the use of these we find first an angle, and then the remaining side.

Ex. 1. In the triangle CDE (fig. 24) given the side CE 274.703, the side DE 589.101, to find by means of the table the hypothenuse CD. Ans. 650.

Ex. 2. In the triangle CDE (fig. 24) given the hypothenuse CD 745, the side CE 427.317, to find, in like manner, the side DE. Ans. 610.

51. From what has been done, it is manifest, that any right angled triangle whatever may be determined by means of a table constructed in the manner, which has been described, provided that the triangle proposed be similar to some one of the triangles of the table, and that two of its parts be given, exclusive of the right angle.

SECTION IX.

OF COTANGENTS AND COSECANTS.

52. For the sake of convenience the table now constructed is usually still further extended, by writing down against the several angles in the table the tangents and secants of their complements.

53. If from the extremity of the radius BC (fig. 26), we draw the tangent BF to meet the secant CG, then, since the arc EB is the complement of the arc AE, BF will be the tangent, and CF the secant of the complement of the arc AE; BF may therefore for the sake of brevity, be called the *cotangent*, and CF the *cosecant* of AE.

54. The cotangent and cosecant of the arc AE (fig. 26), evidently belong to a triangle distinct from that, to which the tangent and secant are found; the former of these, however, may be deduced from the latter. Indeed, on account of the similar triangles CAG, CBF, we have GA : AC : : CB : BF, that is, tang AE : R : : R : cot AE, hence

$$\text{cot AE} = \frac{R^2}{\text{tang AE}}$$

In order to find the cotangent of an arc, when its tangent is given; *we divide the square of the radius by the tangent of the arc.*

Again, to find FC, we have GA : GC : : BC : FC, that is, tang AE : sec AE : : R : cosec AE, hence

$$\text{cosec AE} = \frac{\text{sec AE} \times R}{\text{tang AE}}$$

To find therefore the cosecant of an arc, when its tangent and secant are given; *we multiply the secant of the arc by radius, and divide the product by its tangent.*

55. To show the use, which may be made of the cotangents and cosecants, let there be given in the triangle ABC (fig. 27) the side AC 6 yds, the angle at B 40°, to find the remaining side and the hypothenuse. From the point A with an extent equal to one yard describe the arc MK; draw HM perpendicular to AC: AHM will be the tabular triangle similar to the triangle proposed.

1. To find BC we have

AM : HM : : AC : BC
R : Cot B : : AC : BC
100000 : 1.9175 : 6 : 7.1505

2. To find AB, AM : AH : : AC : AB
R : Cosec B : : AC : AB
100000 : 1.55723 : : 6 : 9.3433

3. Given AC 6, and AB 9.3433, to find the angle B.

56. In employing the cotangent and cosecant of B in the preceding example, it is evident, that we have merely used the tangent and secant of A; the work is of consequence the same, as if we had found by subtraction the angle A, and employed the tangent and secant of this angle.

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SECTION X.

MISCELLANEOUS REMARKS ON THE TRIGONOMETRICAL TABLES.

57. In the preceding articles we have shown the manner, in which a set of trigonometrical tables may be formed. Various methods of abridging the labor of calculation will naturally occur in practice. It is sufficient at present to remark, that from the nature of sines and cosines, tangents, and co-

tangents &c. if the sines, cosines, tangents &c. have been calculated to 45° inclusive, the remainder of the series may be considered as known.

58. The several triangles of the tables for a given angle may be exhibited in connection with each other. In the triangle ABC (fig. 28) if we consider AC radius, BC will be the cotangent and AB the cosecant of the angle at B; if then with an extent equal to AC we draw the arc DF, and from the points D and F the lines DE, FG perpendicular to BC, then FG will be the tangent and BG the secant of the angle at B; DE moreover will be the sine and BE the cosine of the same angle, and the triangles ABC, GBF, DBE will be the tabular triangles for the angle at B.

59. The radius, it will be observed, assumes a different position in each of these triangles; in the triangle DBE, which contains the sine and cosine of the angle at B, it forms the *hypothenuse*; in GBF, which contains the tangent and secant of the angle at B, it becomes the *side adjacent* to this angle, and in ABC which contains the cotangent and cosecant of the same angle it becomes the *side opposite.* This circumstance then must evidently be taken into consideration in calculating a proposed triangle by means of the tables. If the hypothenuse of the proposed triangle be made radius, or if it be compared with the radius of the tables, the *sines* and *cosines* must be employed in the calculation of the triangle; if the side adjacent a given or required angle be made radius, the *tangents* and *secants* must be employed in the calculation; and if the side opposite the given or required angle be made radius, the *cotangents* and *cosecants* must be employed.

60. The numbers in the tables, the construction of which has now been explained, are usually denominated *natural*

sines, tangents, &c. By means of these numbers, it has been seen, that the sides and angles of any proposed triangle, which is similar to some one of the triangles of the tables, may be accurately determined ; the calculations, however, must be made by the tedious processes of multiplication and division. To avoid this inconvenience, another set of tables may be constructed by writing down against each of the angles the logarithm of its natural sine, tangent, &c. ; of consequence, in the use of the numbers of such a set of tables, addition and subtraction will take the place of multiplication and division.

61. The logarithmic sines, tangents, &c. are commonly called *artificial* sines, tangents, &c. to distinguish them from the natural sines, tangents, &c. In comparing these two sets of tables, one circumstance will require consideration. The radius, to which the natural sines, &c. are calculated, is unity. The secants and a part of the tangents are therefore greater than a unit ; while the sines and another part of the tangents are less than a unit ; of consequence, when the logarithms of these are taken, some of the indices will be *positive*, and others *negative*. To remedy therefore the inconvenience, which would result from the occurrence of both positive and negative indices in the same set of tables, ten is added to each of the indices, by which they are all rendered positive. Thus the natural sine of one minute is 0.00029, or more accurately 0.000290888 ; the logarithm of this is $\bar{4}.46373$; but the index by the addition of 10 becomes (10 — 4) or 6 ; of consequence the logarithmic sine of one minute is 6.46373. In like manner the logarithmic sine of two minutes is 6.76476, and so of the rest.

62. The learner may now solve, by the aid of the logarithmic sines &c. the following cases.

Ex. 1. Given the hypothenuse of a right angled triangle 275, and the angle at the base 57° 23′; to find the remaining parts.

1. To find the perpendicular we have

Radius	10.000000
Is to the hypothenuse, 275,	2.439333
As sine 57° 23′,	9.925465
To the perpendicular, 231.63.	2.364798

2. In like manner we find the base 148.23.

Ex. 2. Given the base of a right angled triangle, 200 feet, and the adjacent angle 47° 30′, to find the perpendicular.
Ans. 218.26 feet.

Ex. 3. Given the hypothenuse 480, and the perpendicular 384, to find the angle opposite the perpendicular.
Ans. 53° 8′.

Ex. 4. Given the base 195, and the perpendicular 216, to find the hypothenuse. Ans. 291.

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SECTION XI.

SOLUTION OF OBLIQUE ANGLED TRIANGLES.

63. From what has been said, it is evident, that, in order to calculate a triangle by means of the tables, we must be able to construct from the numbers in the tables a triangle similar to the one proposed for calculation.

Let there be given the oblique angled triangle ABC (fig. 29, A), it is proposed to construct from the numbers in the tables a triangle, which shall be similar to this. Having circumscribed a circle about the given triangle, from the centre D of this circle draw the radii DA, DB, DC; with the radius D*b*, equal to that of the tables, describe the circle *abc;* draw the chords *ab*, *bc*, *ac* joining the points of section *a*, *b* and *c* ; since *Da* is equal to *Db*, the lines DA, DB are cut proportionally at the points *a* and *b* ; *ab* is therefore parallel to AB ; for a similar reason, *ac* is parallel to AC, and *bc* to BC ; the triangle *abc* is therefore similar to the triangle ABC (Geom. B. IV). Upon the side *ab* of the triangle *abc* let fall from D the perpendicular D*e* ; since *aD* is equal to the radius of the tables, *ae* will be the sine of the angle *aDe*, and *be* the sine of the angle *bDe* ; produce *De* to meet the circumference of the circle *abc* in *h* ; since the arc *ah* is equal to the arc *hb* the angle *aDe* is equal to the angle *bDe* ; the side *ab* of the triangle *abc* is therefore equal to twice the sine of the angle *aDe*, or since the angle *aDe* is equal to the angle *acb* (Geom. B. III), the side *ab* is equal to twice the sine of the angle *acb* ; in like manner *bc* is equal to twice the sine of the angle at *a*, and *ac* to twice the sine of the angle at *b* ; but the angles *a*, *b* and *c* in the triangle *abc* are equal respectively to the angles A, B and C in the triangle ABC ; the sides *ab*, *bc*, *ac* in the triangle *abc* are therefore equal respectively to twice the sine of the angles C, A and B opposite the homologous sides AB, BC, AC in the triangle ABC.

With respect to the side *ab* in the case, where the angle C is greater than a right angle (fig. 29, B) ; since from the definition of a sine, an angle will have the same sine with its supplement to two right angles, *ae*, the sine of the angle *aDe*, will also be the sine of the angle *aDk* ; and the side *ab* will be equal to twice the sine of the angle *aDk* ; but the angle *aDk*, or, which is the same thing, the angle ADE is equal to the angle ACB ; the side *ab* is therefore equal, as before, to twice the sine of the angle at C. From the numbers in the tables we may then always construct a triangle similar to any oblique angled triangle, which may be proposed for calculation, by taking for each of the sides in the required triangle twice the sine of the angle opposite the homologous side in the triangle proposed. The tables, therefore, which at first appear limited to the solution of right angled triangles, may evidently be applied to the solution of triangles of whatever kind.

64 We proceed to the investigation of rules for the calculation of plane triangles of whatever kind by means of the tables. The similar triangles ABC, *abc* (fig. 29) give AB : BC : : *ab* : *bc* &c., that is, AB : BC : : 2 sin C, 2 sin A, or, sin C : sin A ; a proportion, which, it is easy to see, is of general application, and which may be enunciated, as follows; *in any triangle whatever, the sines of the angles are to each other as the sides opposite to these angles.*

65. Case 1. To apply this principle, let there be given in the triangle ABC (fig. 30) the side BC 70 yds, the angle A 86^6, and the angle C 45°, to find the remaining parts.

1. To find AB,

sin A 86°	9.998941
BC 70 yds	1.845098
sin C 45°	9.849485
	11.694583
AB 49.62 yds	1.695642

2. In like manner, AC is found equal to 52.96 yds.

Case 2. Given two of the sides, viz. AB 49.62 yds, BC 70 yds, and the angle A opposite BC 86°, to find the remaining parts.

1. To find the angle at C.

BC 70 yds	1.845098
sin A 86°	9.998941
AB 49.62 yds	1.695642
	11.694583
sin C 45°	9.849485

2. Having obtained the angle C, the side AC is found as in the preceding case.

3. Given in the triangle ABC (fig. 13) the side AB 72 feet, the side AC 41.5 feet, and the angle B 30°, to find the angle C,

AC 41.5	1.618048
sin B 30°	9.698970
AB 72	1.857333
	11.556303
sin C 60° 10′	9.938255

In this last example the solution (art. 16) is ambiguous; since, however, the angles ACB, AC′B are supplements, the one of the other, and the sine is the same for an angle, and its supplement; if the angle 60° 10′ first obtained does not answer the conditions of the question, we then take its supplement 119° 50′ for the angle C, which is sought.

66. From the preceding examples, it appears, that we may always solve a triangle by the rule just obtained, 1. *when a side and two of the angles are given;* 2. *when two of the sides and an angle opposite to one of them are given.*

67. In the triangle ABC (fig. 30) let there now be given the sides AC and BC, and the angle C contained between them, to find the remaining parts. This case, it will be perceived, does not admit of solution by the preceding rule; a new one must therefore be sought. Having described a circle with a radius equal to that of the tables (fig. 31), draw the diameter AM; let the arc AB, set off from A, be supposed equal to the number of degrees, contained in the angle at C in the proposed triangle; and also the arc AC to the number of degrees in the angle at B; from B draw the chord BD perpendicular to AM; from C draw CP perpendicular and CF parallel to AM; join BF, FD; from F, with the radius of

the circle ABD equal to that of the tables, draw the arc IGK meeting CF in G, and, at the point G, draw HL perpendicular to CF ; the line GL is the tangent of the angle GFL, or which is the same thing, of CFD, and GH is the tangent of GFH, or CFB ; and the angles CFD, CFB having their summits in the circumference are measured by half the arcs CD, CB, on which they stand ; but CD is the sum of the two arcs AB, AC, and CB is the difference of these two arcs ; whence GL is the tangent of the half sum, and GH the tangent of the half difference of the arcs AB, AC ; but the arcs AB, AC, by hypothesis, are respectively the measures of the angles A and B in the proposed triangle ABC (fig. 30) ; GL is therefore the tangent of the half sum and GH the tangent of the half difference of the angles A and B ; in like manner it may be shown, that DE is the sum, and BE the difference of the sines of the same angles A and B : wherefore on account of the parallel lines BD, HL we have DE : BE : : LG : GH, or putting for these lines what they severally denote

$$\sin A + \sin B : \sin A - \sin B : : \text{tang}\,\frac{A+B}{2} : \text{tang}\,\frac{A-B}{2};$$

but by art. 64 we have sin A : BC : : sin B : AC, hence

$$\sin A + \sin B : \sin A - \sin B : : BC + AC : BC - AC :$$

$$\text{whence } BC + AC : BC - AC : : \text{tang}\,\frac{A+B}{2}\ \text{tang}\,\frac{A-B}{2};$$

a proportion, which we may thus enunciate : *the sum of two sides of a triangle is to their difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

68. Ex.1. Let AC (fig. 30) be 52. 96 yds, BC 70 yds, and the angle C 45° : it is required to find the remaining

parts. Subtracting the angle C 45° from 180°, and dividing the remainder by 2, we have

$\frac{A+B}{2} = 67° \, 30'$, and to find $\frac{A-B}{2}$ we have

BC + AC	122.96	2.089763
BC — AC	17.04	1.231470
tang $\frac{A+B}{2}$	67° 30′	10.382776
		11.614246
tang $\frac{A-B}{2}$	18° 29′ 57″	9.524483

Having obtained the half difference between A and B, the greater angle A is found by adding the half difference to the half sum, and the less angle B by subtracting the half difference from the half sum. The angles being thus obtained, the remaining side may be calculated as before.

Ex. 2. Given two sides of a plane triangle 450 and 540, and the included angle 80°, to find the remaining parts.

Ans. Angles 56° 11′, 43° 49′, and the side 640.08

Ex. 3. Given two sides of a plane triangle 76 and 109, and the included angle 101° 30′, to find the remaining parts.

Ans. Angles 30°57′ 30″, 47° 32′ 30″, and side 144.8.

69. The preceding rules do not include the case, in which the three sides are given to find the angles. To obtain an expression for this case, in the triangle ABC (fig. 32) let fall from the point B, BD perpendicular to AC, and from the same point, with a radius equal to the side BC, describe the circumference CEHF ; extend the side AB, until it meets the circumference in E ; then, since AE and AC are drawn from the same point A without the circle, AC : AE : : AG : AF (Geom. B.IV); but AE is equal to AB + BC, and AG is equal to AB—BC; AF moreover is equal to AD—DC; therefore AC : AB + BC : : AB — BC : AD — DC. If then

upon the longest side of a triangle we let fall a perpendicular from the opposite angle, we have the following proportion ; *as the longest side of a triangle is to the sum of the two other sides, so is the difference of these last to the difference of the segments made by the perpendicular.*

70. Ex. 1. Given the three sides of a triangle ABC (fig. 33), viz. AB 66 feet, AC 75, and BC 34 feet, to find the angles. By the preceding rule

AC 75	1.875061
AB + BC 100	2.000000
AB — BC 32	1.505150
	3.505150
AD — DC diff. seg. 42.67	1.630089

Adding half the difference of the segments to half the sum, we obtain the greater segment AD 58.83 feet ; subtracting the half difference from the half sum, we obtain the less segment DC 16.17 feet.

Then to find the angle A, AB : R : : AD : cos A 26° 57′ ; and to find the angle C, BC : R : : DC : cos C 61° 36′.

Ex. 2. The sides of a plane triangle are 40, 34 and 25 feet respectively ; required the angles.

Ans. 38° 25′ 20″, 57° 41′ 25″, 83° 53′ 15″.

Ex. 3. The sides of a plane triangle are 390, 350 and 270 feet respectively ; required the angles.

Ans. 42° 21′ 57″, 60° 52′ 42″, and 76° 45′ 21″

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SECTION XII.

MENSURATION OF HEIGHTS AND DISTANCES.

71. One of the most simple applications of Trigonometry is to the determination of the heights and distances of objects, when, by reason of intervening obstacles, a direct meas-

urement cannot be made, or where it cannot be made with convenience.

Thus, let it be required to find the distance between the points A and B (fig. 30), which, by reason of some obstacle, we cannot measure directly.

The distance AB, it is evident, can be found, if we make it a side in a triangle, and determine a sufficient number of the remaining parts of the triangle for its calculation. Let us then, beginning at B, measure in some convenient direction a straight line BC of a length sufficient for our purpose, and at the extremities B and C of this line, measure the angles ABC, BCA. There will then be given three parts in the triangle ABC, by which the required part or distance AB sought may be found.

Ex. Suppose BC, the distance measured, 400 yards, the angle ABC 73° 15′, the angle BCA 68° 2′; what is the distance between the points A and B.

Subtracting the sum of the angles ABC, BCA from two right angles we have BAC = 38° 43′, then

sin BAC : BC : : sin BCA : AB = 593.09 yds. *Ans.*

The line BC which we have measured, forming the basis of the work, is called a *base line.*

For the measurement of lines we employ any of the usual linear measures, as the foot, yard, &c.

The instrument commonly employed for the measurement of the angles required is called a *Theodolite.*

There are two species of angles to be measured, 1°, those in a horizontal plane, as in the preceding example, and which are, therefore, called *horizontal* angles. 2°, those in a vertical plane and which are called *vertical* angles.

The instrument is adapted to the measurement of both these species of angles. Its two principal parts are, therefore, a graduated circle, which by the means provided may be placed in a horizontal plane for the measurement of *horizontal* angles, and another graduated circle or semi-circle placed at right angles to this, for the measurement of *vertical* angles.

The following are a specimen of the problems which most frequently occur in practice.

PROBLEM I.

72. To determine the altitude of a steeple or other object situated on a horizontal plane.

From the bottom of the steeple (fig. 37) measure in a direct line upon the horizontal plane, any convenient base BA ; at the extremity A of this line measure the angle BAC, comprised between the line BA and an imaginary line, drawn from the point A to C the top of the steeple ; then in the triangle ABC the side AB and all the angles will be known ; whence to find BC, we have

$$\sin ACB : AB :: \sin BAC : BC$$

or otherwise $$R : \text{tang } BAC :: AB : BC$$

The angle BAC situated in a plane perpendicular to the horizon, and comprised between a horizontal line AB and an ascending line AC, is called an angle of *elevation.*

Ex. 1. Let the base line AB be 200 feet, and the angle of elevation BAC be 47° 30′ ; required the height of the steeple.
Ans. 218.26 feet.

Ex. 2. Let AB be 384 feet, and the angle at A 36° 52′ ; required the height of the steeple. Ans. 288 feet.

PROBLEM II.

73. To determine the perpendicular height of a cloud or other object above a horizontal plane.

At two convenient stations A and B on the same side of the object (fig. 38) or on opposite sides (fig. 39) and in the same vertical plane, let two observers take at the same time the angles of elevation CAB, CBD, and let the distance BA be measured; then the exterior angle of a triangle being equal to the two interior and opposite, subtracting (fig, 38), the angle CAB from CBD, we have the angle ACB ; whence

sin ACB : AB : : sin CAB : BC ;

then in the triangle CBD we have, to find CD the height required R : BC : : sin CBD : CD ;

or (fig. 39) subtracting the sum of the angles at A and B from two right angles, we have the angle ACB ; then to find one of the remaining sides in the triangle ACB, AC for example, we have sin ACB : AB : : sin ABC : AC ;

and in the triangle ADC, to find CD we have

R : AC : : sin CAD : CD.

Ex. 1. Let the angles of elevation (fig. 38) be 31° and 46° respectively, and the base AB, 100 yards : what is the height of the cloud ? Ans. 143.14 yds.

Ex. 2. Let the angles of elevation (fig. 39) be 53° and 79° 12′ respectively, and the distance AB 100 rods ; what is the altitude of the cloud ? Ans. 105.89 rods.

PROBLEM III.

74. To find the height of an inaccessible object standing on a horizontal plane.

At two stations A and B in the same vertical plane passing through the top of the object (fig. 40), take the angles of elevation CAB, CBD, and measure the distance AB, then in the triangle CAB we have

sin ACB : AB : : sin CAB : CB

whence to find CD the altitude required, we have in the triangle CBD R : CB : : sin CBD : CD

Ex. 1. Let the angles of elevation be 32° and 58°, and the base line 100 yards; required the height of the object.

Ans. 102.51 yards.

Ex. 2. Let the angles of elevation be 40° and 60° and and the base line 100 feet; required the height of the object.

Ans. 162.75 feet.

PROBLEM IV.

75. To find the height of an accessible object standing on an inclined plane.

From the bottom of the object (fig. 41) measure in the same direct line any two distances AB, BD; at B take the angle CBA, and at D the angle CDA; then in the triangle CDB we have $\sin DCB : DB :: \sin CDB : BC$

BC being thus determined, we have in the triangle CBA

$$BC + BA : BC - BA :: \text{tang } \tfrac{1}{2}(A + C) : \text{tang } \tfrac{1}{2}(A - C)$$

having found by means of this proportion one of the remaining angles in the triangle CBA, BCA for example, we have, to find the height required

$$\sin BCA : BA :: \sin ABC : AC$$

Ex. Let the distances AB, BD be 40 and 60 feet, the angle at B 41°, that at D 23° 45′; required the height of the object. Ans. 57.624 feet.

PROBLEM V.

76. To find the distance between two objects inaccessible by reason of an intervening river, or other impassable barrier, the objects being both on the same level.

At two stations A and B (fig. 42) also on a level, we take the angles CAD, DAB, CBA, DBC and measure the distance AB; then in the triangle ACB we have

$$\sin ACB : AB :: \sin CAB : CB$$

and in the triangle ABD

$$\sin ABD : AB :: \sin DAB : DB$$

then in the triangle CBD we have

$$BC + BD : BC - BD :: \text{tang } \tfrac{1}{2}(D + C) : \text{tang } \tfrac{1}{2}(D - C)$$

having found by means of this proportion the angle BCD for example, we have finally for the distance sought

$$\sin BCD : BD :: \sin CBD : CD$$

Ex. Let the angles in the order stated above be 37°, 58° 20′, 53° 30′, 45° 15′, and the base line 300 yards. What is the distance between the two objects? Ans. 479.79 yards.

PROBLEM VI.

77. The height of a tower being given to find the horizontal distance between two objects, situated on the same level with the tower and in a direct line from the bottom of it.

Take the angles EAD, EAC (fig. 43) contained between the line EA parallel to the horizon and lines drawn from A, the top of the tower, to the objects D and C respectively; then in the triangle DAB all the angles and the side AB will be known, from which DB may be found; in like manner in the triangle ACB all the angles and the side AB will be known from which CB may be found; the value of these being determined, the latter subtracted from the former will give the distance sought.

An angle EAD comprised between a line parallel to the horizon and a descending line is called an angle of *depression*.

Ex. Let the height of the tower be 120 feet, the angle of depression of the nearest object 57°, that of the most remote 25° 30′. What is the distance between the two objects?
Ans. 173.65 feet.

PROBLEM VII.

78 To find the altitude of a hill above a horizontal plane which passes through a given point on the side of another hill opposite.

At the given point B (fig. 45) take the angle of elevation CBD of the top of the hill, the altitude of which is sought, from B measure a base line BA directly up the other hill, and at a A take the angle of elevation CAE and the angle of depression EAB; then in the triangle CAB all the angles will be given and the side AB to find CB; and CB being found, we shall have in the triangle CBD all the angles and the side CB to find CD the altitude sought.

Ex. Let the angle CBD be 5° 52′, the base line AB 642 yards, and the angles CAE, EAB 3° 59′, and 39′ respectively. What is the altitude of the hill ? Ans. 161.3 yards.

PROBLEM VIII.

79. To find the distance between two inaccessible objects, both of which can be seen at the same time from one point only.

Let D (fig. 46) be the point from which both objects can be seen at the same time; in any convenient directions take a station C, where A can be seen, and a station E, where B can be seen, and measure the distances CD, DE; at D take the angles ADC, ADB, BDE, and at C and E the angles ACD, BED; then in the triangles ACD, BDE we shall be able to determine the sides AD, DB; and these being found, we shall have in the triangle ADB data sufficient to determine AB the distance sought.

Ex. Let CD be 200 yards, the angle ADC 89°, ACD 50° 30′; let DE also be 200 yards, the angle BDE 54° 30′, BED 88° 30′, and ADB 72° 30′ ; required the distance between the two objects. Ans. 345.5 yards.

PROBLEM. IX.

80 To find the altitude of an inaccessible object situated on an elevation, the observer being on a horizontal plane.

Having assumed a station B (fig. 47) upon the horizontal plane, measure in a direct line from the object a base line BA, at B take the angles of elevation DBE, CBE, and at A the angle of elevation CAE ; in the triangle CAB we determine the side CB, then in the triangle CBE we find the remaining sides CE and BE ; this being done, in the triangle DBE we find the side DE ; subtracting next DE from CE already found we have the altitude CD sought. CD may also, it is evident, be found by means of the triangle CBD without calculating CE and DE.

Ex. Let the angle DBE be 40°, CBE, 51°, the base line BA 100 yards, and the angle CAE, 33° 45′, to find the altitude of the object. Ans. 46.67 yards.

PROBLEM. X.

81. To determine the distance of an inaccessible object, when an instrument for measuring angles cannot be procured.

Let E be the object (fig. 50) and AE the distance sought; from the station A in the direction EA, measure any convenient distance AC ; assume another station B, and in the direction EB measure any convenient distance BD ; measure also the distances AB, AD and BC. In the triangles ACB, ABD we shall have the sides given to determine the angles, and these being found, those of the triangle AEB will be known ; then in the triangle AEB we have the angles and a side to find AE the distance sought.

Ex. Let AB be 500, AC 100, CB 560, BD 100, and AD 550 yards; required the distance AE.

Ans. 536.25 yards.

QUESTIONS AND PROBLEMS FOR PRACTICE.

82. From what has been done the learner will see the manner in which the principles are applied to the object proposed. We subjoin as an exercise a few additional questions and problems.

1. A line 27 yards long will exactly reach from the top of a fort to the opposite bank of a river, known to be 23 yards broad; what is the height of the wall? Ans. 42.42 feet.

2. The height of an object on a horizontal plane is 200 feet, and its angle of elevation at the place of the observer is 42° 30′; what is his distance from the object?

Ans. 218.26 feet.

3. A ladder 193.55 feet long is placed against a wall in an oblique position, and reaches to a point 75.83 feet from the ground; what is the angle at which it is inclined to the ground? Ans. 23° 3′ 55″.

4. From the edge of a ditch 18 feet wide, surrounding a fort, I took the angle of elevation of the top of the wall and found it 62° 40′; required the height of the wall, and the length of a ladder necessary to reach from my station to the top of it. Ans. Height 34.82, length 39.2 feet.

5. A ladder 125 feet long is placed against a wall so that the angle at the bottom is double the angle at the top; how high up the wall does it reach and how far distant from the wall is its foot? Ans. 108.25, and 62.5 feet respectively.

6. A person attempts to swim across a river, whose breadth is 329 yards. How far from the place immediately opposite to his departure does he arrive, if the stream forces

him to swim twice as far as he would have done, had there been no current? Ans. 569.9 yards.

7. From the top of a tower, 143 feet high, by the sea side, I observed that the angle of depression of a ship's bottom, then at anchor, was 55°; what was its distance from the bottom of the wall? Ans. 100.13 feet.

8. Two ships of war wishing to ascertain their distance from a fort, sail from each other a distance of half a mile, when they find that the angles formed between a line from one to the other, and from each to the fort, are 85° 15′, and 83° 45′. What are their respective distances from the fort?

Ans. 4584.52, 4596.10 yards.

9. Wanting to know the breadth of a river, I measured 100 yards in a right line close by one side of it, and at each end of this line I found the angles subtended by the other end and a tree close by the other side of the river to be 53° and 79° 12′. What is the perpendicular breadth?

Ans, 105.89 yards.

10. Being on one side of a river and wanting to know the distance to a house, which stood on the other side, I measured 200 yards in a right line by the side of the river, and found that the two angles at each end of this line formed by the other end and the house were 73° 15′ and 68° 2′; what was the distance between each station and the house?

Ans. 296.54 and 306.19 feet.

11. A gentleman wishing to ascertain the distance between two trees A and B, which could not be directly measured on account of a pool, that occupied the intermediate space, assumed a station C, from which both could be seen, and found by measurement the distance AC 7.35 chains, the distance BC 8.4 chains, and the angle ACB 55° 40′. What is the distance of the trees A and B. Ans. 7.412 chains.

12. From the top of a tower, whose height is 108 feet,

the angles of depression of the top and bottom of a vertical column standing in the horizontal plane are found to be 30° and 60° respectively; required the height of the column.

Ans. 72 feet.

13. From a window B near the bottom of a house supposed to be on a level with a monument CD, the angle of elevation CBD of the top of the monument being 40°, and from another window A 18 feet higher the angle of elevation CAE being 37° 30′, it is required to find the height of the monument CD and its distance BD.

Ans. CD 210.44 feet, and BD 250.79 feet.

14. Wanting to know the height and distance of an object on the other side of a river, and on a level with the place where I stood, close by the side of the river; not having room to go backward on the same plane, on account of the immediate rise of the bank, I placed a mark where I stood and measured in a direct line from the object up the hill a distance of 132 yards; I then found the angle of depression of the mark by the river's side 42°, that of the bottom of the object 27°, and of its top 19°; required the height of the object, and the distance of the mark from its bottom.

Ans. Height 28.63, distance 75.25 yards.

15. It is required to find the height of a castle AB, situated on an eminence by the sea shore, above the level of the sea and its horizontal distance from a ship S at anchor, the angles of depression, EAS, FBS being given equal respectively to 4° 52′, and 4° 2′, and the height of the castle being 54 feet.

Ans. Distance 3690 feet, and height 314 feet.

16. At a station A at the bottom of a hill, I took the angle of elevation of the top of an object D on the summit of the hill, 38°, and measuring directly up the hill a distance of 676.47 feet to another station B, I took at this station the

angle of elevation of the top of the object D, 46°, and the angle of depression of the first station A, equal to the angle of elevation of the second station from the first, 27° 30′ 25″; what is the height of the object D above the level of the first station A? Ans. 949.19 feet.

17. Suppose the object CD (fig. 49) to stand upon a horizontal plane ABD and that AB is equal to 250 yards, and that the angles at its extremities are known, viz. CAB 56° 46′, CBA 62° 54′, DAC 6° 40′, DBC 7° 6′. What is the height CD and the two distances AD and BD?

Ans. AD 254.989, BD 238.814, and CD 29.745 yds.

18. To ascertain the distance between two places, the angle which they subtended at a point equally distant from both was observed, and again at another point equally distant from both; and the distance between the points of observation was measured. To determine from these data the distance of the places, 1° when the points of observation are both on the same side of the places, 2°, when they are on opposite sides.

19. At a given distance from an obelisk, whose height is known, a colossal statue on the top of it subtends the same angle as the observer, when seen from the base of the obelisk. Supposing the obelisk and the observer on the same horizontal plane, and the height of the observer known, show how the height of the statue may be determined.

20. Two spectators at two different stations on a horizontal plane, the distance between which is known, observe at the same instant a balloon, which rises in the air at a uniform rate, and after a given interval of time again observe it; what are the observations necessary to determine the two elevations of the balloon and its rate of moving, and how shall the data be applied to this purpose, 1°, when it ascends in a perpendicular line and the stations are not in the

same vertical plane with it, 2°, when it ascends in an oblique line and the stations are in the same vertical with it?

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SECTION XIII.

LINE OF NATURAL SINES, TANGENTS, &C.

83. Let the quadrant AB (fig. 35) be divided into equal portions, of a degree each, for example; and let lines be drawn, parallel to AC, from each point of division in the quadrant AB to the line CB. The several distances from C to the points where these parallels meet the line CB, will be equal respectively to the lengths of the sines for each degree in the quadrant AB. Thus the extent from C to 10 will be equal to the sine of 10°, from C to twenty to the sine of 20°, and so on. The line CB, divided in this manner, is called *a line of sines.*

In like manner if through each of the points of division in the quadrant AB secants be drawn to meet the indefinite tangent BD, the several distances from B to the points of division in BD will be equal respectively to the tangents for each degree in the quadrant AB, and the line BD will be a *line of tangents.*

If moreover the different distances from the centre C to each of the divisions on the line of tangents BD be transferred to the line CE, the line CE will be a *line of secants.* Thus B 20 is the tangent, and C 20 the secant of 20°.

84. A single example will show the use, which may be made of these lines.

In the triangle ABC (fig. 30), given the side BC 70 yds, the angle A 86°, and the angle C 45°, to find AB.

By art. 65, sin A 86° : sin C 45° : : BC : AB. In order therefore to determine the value of AB by the line of sines, we measure by means of a convenient scale of equal parts, the extent from C to 86 on the line of sines (fig. 35), and also the extent from C to 45 ; we then have the relative values of the first and second terms in the above proportion.

Let the sine of A determined in this manner be 332, and the sine of C 235, then

332 : 235 : : 70 : 49.5 the side AB, &c.

The method of solving triangles, which has now been exhibited, is obviously the same, as that by the table of natural sines, &c, with the exception that the values of the sines in the present case are determined by geometrical construction.

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SECTION XIV.

LINE OF LOGARITHMIC SINES, TANGENTS, &C.

85. The logarithms of numbers may be conveniently represented by lines. Thus, since the logarithm of 10 is 1 and the logarithm of 100 is 2, if a line of one foot be made to represent the logarithm of 10, a line of two feet will represent the logarithm of 100.

For the purpose of forming a logarithmic scale let the line *ab* (fig. 36), intended for the scale and taken of any length at pleasure, represent the logarithm of 100 ; then 100 will stand at the end of the scale ; and 1, the logarithm of which is 0, will stand at the beginning of the scale. For the intermediate numbers 1, 2, 3, 4, &c, it is easy to see, that we have merely to set off in order from the beginning of the scale the several lengths, which shall be equal respectively to the logarithms of these numbers, when *ab* is equal to the logarithm of 100. The logarithm of 100, taken from a common table of logarithms but extending to three places only of decimals, is 2.000, or, omitting the separatrix, 2000. Under the same circumstances the logarithm of 2 is 301, and the logarithm of 3 is 477, &c. Let it be supposed, that the line *ab* is divided into 2000 equal parts ; then, as *ab* has been made to represent the logarithm of 100, 301 of these parts will represent the logarithm of 2. To determine therefore the place of the number 2 on the scale, we set 301 of the parts, into which *ab* is supposed to be divided, from 1 to 2 ; to determine the place of 3 we set off in like manner 477 of the same parts, and so of the rest. Proceeding in this manner we determine the primary divisions of the scale, and to obtain the intermediate divisions we set off in a similar manner the logarithms of the intermediate numbers. Thus, when the logarithm of 100 is 2000, the logarithm of 1.1 is 41, the logarithm of 1.2 is 79, &c. These numbers being set off in order from the beginning of the scale will divide the first primary division into 10 parts. In like manner the remaining primary divisions may be further divided.

86. The logarithmic scale, the construction of which has now been explained, contains the logarithms of numbers from one to a hundred only. The decimal part of the logarithm of any number, it will be recollected, is the same as that of the number, when multiplied or divided by 10 or 100, &c. We may then consider the point under the 1 at the beginning of the scale to represent the logarithm of 1, or 10, or 100, or $\frac{1}{10}$, or $\frac{1}{100}$, &c. Hence the same scale may easily be made to answer for all numbers whatever.

87. From what has been said the logarithm of any given number may with facility be taken from the scale.

Let it be required to take from the scale the logarithms of the following numbers, viz. 75, 43.5, 365, 2450, 10844.

88. Multiplication, division, &c. are performed by the line of numbers on the same principles as by common logarithms.

Multiplication.

1. Let it be required to multiply 6 by 8. Since addition in logarithms takes the place of multiplication, the extent from 1 to 6 added to the extent from 1 to 8 will be equal to the extent from 1 to 48 the product.

To multiply then by the logarithmic line, we take off with the compasses that length of line which represents the logarithm of *one* of the factors, and apply this so as to extend forward from the end of that which represents the logarithm of the *other* factor. The sum of the two will reach to the end of the line, which represents the logarithm of the product.

2. It is required to multiply 9 by 7, 18 by 5, 44 by 63, 120 by 75.

Division.

It is required to divide 56 by 7, 120 by 12, 400 by 50, 2880 by 320.

Proportions.

1. To find the fourth term of the proportion, in which 2, 4 and 8 are the first three terms.

2. To find the fourth term of the proportion, in which 15, 75 and 40 are the first three terms.

3. Let 4324, 540.5 and 2560 be the three first terms of a proportion ; required the fourth term.

89. We proceed to construct a line of logarithmic sines, that shall correspond with the line of numbers. For this purpose let the line *cd*, intended for the scale, be taken (fig. 36) equal to *ab*. In order that this scale may correspond with the line of numbers, it will be obvious, that the difference between the extreme indices on both should be the same. On the line of numbers the difference between

the extreme indices is 2. The logarithmic sine of 0° 34′ 22″ 41‴, is 8.000, and that of 90° is 10.000 the difference of the indices being 2. If then the point *c* at the beginning of the scale be marked for the place of 0° 34′ 22″ 41‴ the point *d* at the end of the scale will be the place for 90°. In order to find the intervening divisions we suppose the line *cd*, in the same manner as has been supposed with respect to *ab*, to be divided into 2000 equal parts. Then, since the difference between the logarithmic sine of 0° 34′ 22″ 41‴ and the logarithmic sine of 1° is 241, the extent from *c* to 1 equal to 241 of the parts, into which *cd* is supposed to be divided, will give the point of division for the sine of 1° ; in like manner the extent from 1 to 2, equal to 542 of the same parts, will give the point of division for 2°, and so of the rest. The scale may also be constructed by marking the point *d* for the sine of 90°, and setting the arithmetical complement of each degree &c. backward from *d* toward *c*.

90. The logarithmic tangent of 0° 34′ 22″ 35″, is 8.000, and that of 45° is 10.000. To form therefore a line of logarithmic tangents corresponding to the line of numbers, let the point *e*, on the line *ef* taken equal to *ab* be marked for the tangent of 0° 34′ 22″ 35‴, and the point *f* for the tangent of 45° ; then, *ef* being supposed also to be divided into 2000 equal parts, the intervening points of division may be marked as before.

For the points of division above 45° the scale should extend much further to the right; but (art. 54) $\frac{\mathrm{R}}{\mathrm{tang}} = \frac{\mathrm{cot}}{\mathrm{R}}$, that is, in logarithms $\mathrm{R} - \mathrm{tang} = \mathrm{cot} - \mathrm{R}$. The logarithmic tangent therefore of an arc below 45° is as much less than that of 45°, as the logarithmic cotangent of this arc is greater than that of 45°. Instead then of extending the scale further to the right, the numbering after reaching 45° may be continued back from right to left, and the same point of division be made to answer for an arc and for its complement.

91. We shall now show the use, which may be made of the logarithmic lines in the solution of triangles.

Ex. 1. In the triangle ABC (fig. 30), given the side AB 40 yds, the side AC 90 yds, and the angle C 20°, to find by means of the logarithmic lines the angle B.

By art. 65, AB : AC : : sin C : sin B; wherefore in logarithms AC — AB = sin B — sin C; hence to find the angle at B, we take in the compasses from the line of numbers the extent from 40 to 90; this extent will reach from 20° on the line of sines to 50° 20′ the answer.

2. Let the angle C equal 55°, the angle B 32°, and the side AB 100 feet to find AC.

3. Given AB 53, BC 13, and the angle B, 120°, to find AC.

92. The scale fig. 36, the construction of which has now been explained, is commonly called *Gunter's scale*, from the name of its inventor. It furnishes a ready method of solving triangles in those cases, where great accuracy is not required. The mode of solution differs from that by the common tables of logarithms in this respect only, that the logarithmic values of the sides and angles are expressed by lines instead of numbers.

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SECTION XV.

TRIGONOMETRICAL ANALYSIS.

93. In the preceding sections we have treated of sines, cosines, tangents, &c. merely in their relation to the calculation of the sides and angles of triangles. The use of these

lines may however be extended to other objects. In conducting the investigations of physical astronomy in particular, important assistance is derived from them ; because as it is only the angular positions of the heavenly bodies, which we observe, all disquisitions concerning the motions, orbits, &c. of these bodies must necessarily involve the sines, cosines, tangents, &c. of angles.

94. In the solution of plane triangles the arcs, which fall under consideration, never exceed a semi-circumference ; and we have occasion to consider merely the numerical values of the sines, cosines, tangents, &c. But in the more extended applications of these lines, the arcs, which occur, assume every possible magnitude, and the sines, cosines, &c. being regarded as *functions* of their corresponding arcs, that is, as analytic expressions, the values of which depending entirely upon those of the arcs may be deduced from them by certain arithmetical operations, we shall have occasion to consider not only the changes that take place in the *numerical values* of the different trigonometrical lines, but also the changes that occur in their *positions* as the arcs to which they belong vary in magnitude.

95. To understand clearly the nature of these changes let us attend to the variations, which take place in the numerical values, and in the positions of the sines, cosines, tangents, &c. of an arc, while it increases from 0 to an entire circumference.

In the circle ABDE (fig. 34) let the diameters AD, BE be drawn perpendicular to each other; the circle will then be divided into quadrants. The angle ACB is called the *first* quadrant, the angle BCD the *second* quadrant, the angle DCE the *third*, and the angle ECA the *fourth* quadrant.

To trace the changes in question, let the radius CF, at first coinciding with CA, be supposed to revolve about the point C, as upon a pivot, and departing from the position CA, so that the arc AF shall have successively all magnitudes whatever from 0 to an entire circumference, and the angle ACF shall increase from 0 to four right angles. In the first quadrant, when the radius CF coincides with CA, so that the arc AF is 0, the sine GF, it is evident, is also 0, while the cosine CG is equal to radius. When the radius CF moves off from CA, the sine GF increases as the point F advances toward B, until, when the point F has arrived at B, the sine GF coincides with CB, and becomes equal to radius. At the point B the arc AF is equal to a quadrant, and the angle ACF to a right angle. Under the same circumstances the cosine CG constantly decreases, until when the point F coincides with B, the cosine becomes equal to 0. In the second quadrant as the point F′ moves on from B to D, the sine G′F′ decreases, and the cosine CG′ increases, until, when the point F′ coincides with D, and the arc AF′ becomes equal to a semi-circumference, the sine G′ F′ is 0, and the cosine CG′ is equal to radius.

It may be remarked in passing, that the lines G′F′, CG′ are respectively the sine and cosine of the arc DF′, and also of its supplement AF′; whence *the absolute magnitude of the sine and cosine of an obtuse angle is the same with that of its supplement.*

In the third quadrant, as the point F″ moves on from D to E, the sine G″F″ increases, and the cosine CG″ decreases, until at the point E, where the arc AF″ becomes equal to three fourths of a circumference the sine is equal to radius and the cosine to 0. From E to A the sine G‴F‴ decreases and the cosine CG‴ increases, until at A, when the whole revolution has been completed, and the arc AF‴ becomes equal to an entire circumference, the sine is reduced to 0, and the cosine becomes equal to radius.

In the first and second quadrants the sines, being all situated on the upper side of the diameter AD, from which they are measured, are considered as *positive*; in the third and fourth quadrants falling below the diameter AD, and assuming in consequence a direction opposite to that, which they had before, they are regarded as *negative*. In the first and fourth quadrants the cosines, which are all situated on the left of the point C, to which they are referred, are considered *positive*; in the second and third quadrants falling on the right of C, and assuming a direction opposite to their former direction, they are regarded as *negative*.

96. Observing in like manner the changes, which occur in the magnitude of the tangents through the course, which has been described, we find, that from A to B the tangents increase continually, as the arc AF increases; at the point B where the arc AF becomes equal to a quadrant, the secant CM coinciding with CB is parallel to the tangent AM, and therefore no longer meets it, so that the arc AB has not properly speaking a trigonometrical tangent. We say, indeed, that the tangent of AB or 90° is *infinite;* but by this expression we mean, that if the difference between an arc and 90° is indefinitely small, the tangent of this arc will be indefinitely great, that is, greater than any assignable quantity. From B to D the tangents decrease, until at D the tangent is 0; from this point they increase, until at E the tangent is again infinite; from E they decrease, and the tangent becomes 0 at A.

With respect to the changes, which occur in the positions of the tangents, from A to B the radius CF produced meets the indefinite tangent at A above the diameter AD; in the first quadrant therefore we consider the tangents as *positive*. From the point B to D, the radius F′C produced no longer meets the indefinite tangent at A above the diameter AD, but below it; in the second quadrant then the tangents are *negative.* Tracing in like manner the changes of position, which take place with respect to the tangents in the remaining quadrants, we find, that in the third quadrant they again become *positive*, and in the fourth *negative.*

In like manner it will be seen that the cotangent is equal to 0 at B and E, and is infinite at A and D. In the first and third quadrants it is *positive*, and in the second and fourth *negative.*

Tracing also the changes which occur in the secants, in the first quadrant the secant is equal to radius at A, and going on to increase through this quadrant it becomes infinite at B; it then decreases through the second quadrant until it becomes equal to radius at D. From this point it again increases through the third quadrant until it becomes infinite at E; from which through the fourth quadrant it decreases and becomes again equal to radius at A.

In the first and fourth quadrants the secants falling above the diameter are regarded as *positive;* falling below in the second and third quadrants they are regarded as *negative.*

The cosecant is equal to radius at B and E. In the first and second quadrants it is *positive*, and in the third and fourth it is *negative.*

Let Q denote a quadrant; then the algebraic signs of the trigonometrical lines determined above will be more distinctly exhibited in the following table.

	First Q.	*Second Q.*	*Third Q.*	*Fourth Q.*
Sine	+	+	—	—
Cosine	+	—	—	+
Tangent	+	—	+	—
Cotangent	+	—	+	—
Secant	+	—	—	+
Cosecant	+	+	—	—

97. We have traced the changes, which occur with respect to the sines, cosines, &c. from the departure of the point F from A until its return to this point again. We may now suppose that at A it commences a second revolution, and regarding as one arc the whole course passed over by the point F from the commencement of its motion, we shall have arcs, that exceed a circumference, and which have the same sines, cosines, tangents &c. as those described in the first revolution.

98. The formulas for the sine and cosine of the *sum*, and the *difference* of two arcs obtained by a geometrical demonstration art. 32, 33 are true whatever the magnitude of these arcs. Representing the arcs by the letters a and b respectively, those formulas, radius being considered as unity, may now be expressed in algebraic language as follows

1. $\sin(a+b) = \sin a \cos b + \sin b \cos a$
2. $\sin(a-b) = \sin a \cos b - \sin b \cos a$
3. $\cos(a+b) = \cos a \cos b - \sin a \sin b$
4. $\cos(a-b) = \cos a \cos b + \sin a \sin b$

99. The four preceding formulas form the basis of trigonometrical analysis. We shall now employ them in the preparation of various other formulas of the greatest utility in the more extended application of the trigonometrical lines. Before making use of them, however, for this purpose, it may be well, in order to see the precise agreement between the analytic and geometrical mode of considering the subject, to show from these expressions the variations, both in sign and magnitude, of the sines and cosines through the first circumference.

Let p denote a semi-circumference; then the cosine of $\frac{1}{2}p$ is 0, and its sine is 1; hence substituting $\frac{1}{2}p$ for a in the preceding formulas, we have

$$\sin(\tfrac{1}{2}p \pm b) = \cos b, \quad \cos(\tfrac{1}{2}p \pm b) = \mp \sin b.$$

To show the absolute value of these expressions, let the arc BF′ (fig. 34) $= b$; since the arc AB $= \frac{1}{2}p$, the arc AF′ $= (\frac{1}{2}p + b)$; then G′F′ the sine of AF′, and also of DF′, is of consequence the cosine of BF′; hence $\sin(\frac{1}{2}p + b) = \cos b$; CG′ moreover the cosine of AF′ is the sine of BF′; hence $\cos(\frac{1}{2}p + b) = \sin b$. If we make BF $= b$, then $\sin(\frac{1}{2}p - b) = \cos b$, and $\cos(\frac{1}{2}p - b) = \sin b$.

With respect to the sign —, which affects the absolute value of $\cos(\frac{1}{2}p + b)$, it signifies, that if we regard as positive the sine and cosine of an arc less than a quadrant, the cosine of an arc greater than a quadrant, but less than a semi-circumference, will be *negative*, while its sine is *positive*.

If we make $b =$ a quadrant, then $\sin(\frac{1}{2}p \pm b) = 0$, and $\cos(\frac{1}{2}p \pm b) = \mp 1$.

Again let $a = p$, then $\sin(p \pm b) = \mp \sin b$, and $\cos(p \pm b) = -\cos b$; if we make $b = \frac{1}{2}p$, then $\sin\frac{3}{2}p = -1$, and $\cos\frac{3}{2}p = 0$.

Lastly let $a = \frac{3}{2}p$, then $\sin(\frac{3}{2}p \pm b) = -\cos b$, and $\cos(\frac{3}{2}p \pm b) = \pm \sin b$.

The absolute values of these last expressions may also be easily verified; the signs show that every arc comprehended between p and $\frac{3}{2}p$ has its sine and cosine negative, while an arc comprehended between $\frac{3}{2}p$ and $2p$ has its sine negative, but its cosine positive.

Expressions for the sines and cosines of multiple arcs.

100. Putting in the expression for $\sin(a + b)$ no. 1 art. 98, $b = a$, $2a$, $3a$, &c., we shall have

$$\sin 2a = 2 \sin a \cos a$$
$$\sin 3a = \sin a \cos 2a + \sin 2a \cos a$$
$$\sin 4a = \sin a \cos 3a + \sin 3a \cos a$$
$$\sin 5a = \text{\&c.}$$

Making the same substitutions in the expression for cos $(a+b)$ no. 3 art. 98, we have

$$\cos 2a = \cos^2 a - \sin^2 a$$
$$\cos 3a = \cos a \cos 2a - \sin a \sin 2a$$
$$\cos 4a = \cos a \cos 3a - \sin a \sin 3a$$
$$\cos 5a = \&c.$$

By the expressions $\sin^2 a$, $\cos^2 a$, &c, we understand the square of the sine of a, square of the cosine of a, &c.

101. To find expressions for sin $3a$, sin $4a$, cos $3a$, cos $4a$ &c. in terms of the simple arc, we substitute in the formulas just obtained the values of sin $2a$, cos $2a$ &c. Thus we have $\sin 3a = 3 \sin a \cos^2 a - \sin^3 a$

$$\sin 4a = 4 \sin a \cos^3 a - 4 \sin^3 a \cos a$$
$$\sin 5a = \&c.$$
$$\cos 3a = \cos^3 a - 3 \sin^2 a \cos a$$
$$\cos 4a = \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a$$
$$\cos 5a = \&c.$$

Expressions for the sines and cosines of half a given arc.

102. In the expression for cos $2a$ art. 100, making $a = \frac{1}{2}a$, we have $\cos^2 \frac{1}{2}a - \sin^2 \frac{1}{2}a = \cos a$ (1)

but art. 27 $\cos^2 \frac{1}{2}a + \sin^2 \frac{1}{2}a = 1$ (2)

whence by subtraction $2 \sin^2 \frac{1}{2}a = 1 - \cos a$

wherefore $\sin \frac{1}{2}a = (\frac{1}{2} - \frac{1}{2}\cos a)^{\frac{1}{2}}$

Adding next the equations 1 and 2. and proceeding in like manner, we obtain $\cos \frac{1}{2}a = (\frac{1}{2} + \frac{1}{2}\cos a)^{\frac{1}{2}}$

Expressions for the products of sines and cosines.

103. Adding and subtracting the four equations, art. 98, and reducing, we obtain

$$\sin a \cos b = \tfrac{1}{2}\sin(a+b) + \tfrac{1}{2}\sin(a-b) \quad (1)$$
$$\sin b \cos a = \tfrac{1}{2}\sin(a+b) - \tfrac{1}{2}\sin(a-b) \quad (2)$$
$$\cos a \cos b = \tfrac{1}{2}\cos(a+b) + \tfrac{1}{2}\cos(a-b) \quad (3)$$
$$\sin a \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b) \quad (4)$$

Expressions for the powers of sines and cosines.

104. 1. Let $b=a$ in nos 3 and 4 of the preceding article, and we have

$$\sin^2 a=\tfrac{1}{2}-\tfrac{1}{2}\cos 2a$$
$$\cos^2 a=\tfrac{1}{2}+\tfrac{1}{2}\cos 2a$$

2. Multiplying the first of the preceding equations by $\sin a$, or to avoid fractions by $4\sin a$, we have

$$4\sin^3 a=2\sin a-2\sin a\cos 2a$$

but art. 103 $\sin(a+b)-\sin(a-b)=2\sin b\cos a$; in this last equation let $a=2a$ and $b=a$, then

$$\sin 3a-\sin a=2\sin a\cos 2a;$$

wherefore by substitution and reduction

$$\sin^3 a=\frac{3\sin a-\sin 3a}{4};$$

and by a process altogether similar, we obtain

$$\cos^3 a=\frac{3\cos a+\cos 3a}{4}$$

3. In like manner, we find

$$\sin^4 a=\frac{\cos 4a-4\cos 2a+3}{8}$$
$$\cos^4 a=\frac{\cos 4a+4\cos 2a+3}{8}.$$

Expressions for the tangents of arcs in terms of the tangents.

105. By art. 45, we have $\tan(a+b)=\frac{\sin(a+b)}{\cos(a+b)}$, whence by substitution $\tan(a+b)=\frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b-\sin a\sin b}$ dividing the numerator and denominator of the second member of this equation by $\cos a\cos b$, we have for the numerator

$$\frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b} = \frac{\sin a}{\cos a} + \frac{\sin b}{\cos b} = \tan a + \tan b,$$

and for the denominator

$$\frac{\cos a \cos b - \sin a \sin b}{\cos a \cos b} = 1 - \frac{\sin a}{\cos a} \times \frac{\sin b}{\cos b} = 1 - \tan a \tan b;$$

whence by substitution and reduction, we obtain

$$\tan (a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \qquad (1)$$

In like manner we obtain $\tan (a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$ (2)

By art. 54, we have $\cot (a + b) = \dfrac{1}{\tan (a + b)}$, whence by substitution

$$\cot (a + b) = \frac{1 - \tan a \tan b}{\tan a + \tan b} \qquad (3)$$

So also

$$\cot (a - b = \frac{1 + \tan a \tan b}{\tan a - \tan b} \qquad (4)$$

Expressions for the tangents and cotangents of multiple arcs in terms of the simple arcs.

106. Putting successively $b = a$, $2a$, $3a$, &c, in nos. 1 and 3 of the preceding article and substituting for factors of multiple arcs, we have

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

$$\tan 4a = \frac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a}$$

$$\tan 5a = \&c.$$

The expressions for the cotangents, will it is easy to see, be merely an inversion of the preceding.

Expressions for the tangents and cotangents of half an arc.

107. By art 48 we have $\tan \frac{1}{2} a = \dfrac{\sin \frac{1}{2} a}{\cos \frac{1}{2} a}$, whence by substitution art. 102,

$$\tan \tfrac{1}{2} a = \frac{(1 - \cos a)^{\frac{1}{2}}}{(1 + \cos a)^{\frac{1}{2}}}$$

multiplying numerator and denominator by $(1 + \cos a)^{\frac{1}{2}}$

$$\tan \tfrac{1}{2} a = \frac{(1 - \cos^2 a)^{\frac{1}{2}}}{1 + \cos a} = \frac{\sin a}{1 + \cos a}$$

The expression for the cotangent $\frac{1}{2} a$, it is evident, will be the inverse of that for the tangent $\frac{1}{2} a$.

Expressions for secants and cosecants.

108. By art. 45 we have $\sec (a + b = \dfrac{1}{\cos (a + b)}$ whence by substitution

$$\sec (a + b) = \frac{1}{\cos a \cos b - \sin a \sin b}$$

dividing both numerator and denominator by $\cos a \cos b$ and substituting we have

$$\sec (a + b) = \frac{\sec a \sec b}{1 - \tan a \tan b}$$

so also
$$\sec (a - b) = \frac{\sec a \sec b}{1 + \tan a \tan b}$$

And by a process altogether similar we obtain

$$\operatorname{cosec} (a \pm b) = \frac{\sec a \sec b}{\tan a \pm \tan b}$$

Making $b = a$ in the expressions for sec and cosec $(a + b)$ we have

$$\sec 2 a = \frac{\sec^2 a}{1 - \tan^2 a}$$

$$\operatorname{cosec} 2 a = \frac{\sec^2 a}{2 \tan a}$$

109. From what has been done, it will be perceived that an indefinite number of trigonometrical formulas may be prepared adapted to the purposes, for which they may be required. We shall close the subject, with the investigation of formulas for finding the angle of a triangle when the sides are given, and also for determining the area of a triangle from different data.

Formulas for the angles of Triangles.

110. In the triangle ABC (fig. 33), let fall from one of the angles, B for example, upon the side AC the perpendicular BD; then (Geom. B. IV) $AB^2 = AC^2 + BC^2 \mp 2\ AC \times DC$, the sign — being used when the perpendicular falls within the triangle, and the sign + when it falls without; but in the right angled triangle BDC, radius being 1, $DC = BC \times \cos C$, wherefore $AB^2 = AC^2 + BC^2 - 2\ AC \times BC \times \cos C$, the sign — being sufficient for the term 2 AC &c. because when the angle C is obtuse its cosine is negative, and consequently changes — into + as is required by the geometrical construction. Hence, employing the letters a, b, c, to represent the sides BC, AC, AB, and deducing the value of cos C, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}$$

But this expression not being well adapted to calculation by logarithms, another is to be sought.

The formula $\sin^2 a = \frac{1}{2} - \frac{1}{2} \cos 2\,a$ gives by substituting $\frac{1}{2}$ C for a, and deducing the value of cos C,

$$\cos C = 1 - 2 \sin^2 \tfrac{1}{2} C\,;$$

comparing the two values of cos C, and deducing that of $2 \sin^2 \frac{1}{2}$ C, we have

$$2 \sin^2 \tfrac{1}{2} C = 1 + \frac{c^2 - a^2 - b^2}{2\,ab} = \frac{c^2 - a^2 - b^2 + 2\,ab}{2\,ab}$$

$$= \frac{c^2 - (a - b)^2}{2\,ab} = \frac{(c + a - b)\,(c - a + b)}{2\,ab}$$

whence

$$\sin^2 \tfrac{1}{2} \mathrm{C} = \frac{(c+a-b)(c-a+b)}{4\,ab} = \frac{\frac{1}{2}(c+a-b)\,\frac{1}{2}(c-a+b)}{ab}$$

but $\frac{1}{2}\,(c+a-b) = \frac{1}{2}\,(c+a+b) - b$, and $\frac{1}{2}\,(c-a+b = \frac{1}{2}\,(c+a+b) - a$, whence representing $c+a+b$ by s and reducing, we have

$$\sin \tfrac{1}{2} \mathrm{C} = \left(\frac{(\frac{1}{2}s-a)\,(\frac{1}{2}s-b)}{ab}\right)^{\frac{1}{2}} \qquad (1)$$

Deducing next the value of cos C from the formula $\cos^2 a = \frac{1}{2} + \frac{1}{2}\cos 2\,a$, we obtain by a process altogether similar

$$\cos \tfrac{1}{2} \mathrm{C} = \left(\frac{\frac{1}{2}s\,(\frac{1}{2}s-c)}{ab}\right)^{\frac{1}{2}} \qquad (2)$$

111. From the formula (1) we obtain the following rule, by which to find an angle of a triangle, when the three sides are given, viz.

From the half sum of the three sides subtract successively those which contain the angle sought; multiply the two remainders together; and divide the product by the product of the sides which contain the angle sought, and the square root of the quotient will be the sine of half this angle.

EXAMPLES.

1. Given the three sides of a triangle equal to 50, 60, and 70 feet respectively to find the angles.

To find the angle contained between the sides 60 and 70, we have $\frac{1}{2}s = 90$, $\frac{1}{2}s - 60 = 30$, $\frac{1}{2}s - 70 = 20$; whence by logarithms

$$\begin{array}{rl}
\log 30 = & 1.477121 \\
\log 20 = & 1.301030 \\
\log 60, \text{ comp.} = & 8.221849 \\
\log 70, \text{ “ } = & \underline{8.154902} \\
2\,)\; & \underline{19.154902} \\
\sin 22^\circ\, 12'\, 28'' = & 9.577451
\end{array}$$

The angle sought will be therefore 44° 24′ 56″.

In like manner the remaining angles may be found.

In the above example we have used the arithmetical complements of the factors of the divisor, in order, Alg. art. 211, that the whole work may be performed by the addition of logarithms.

Ex. 2. In a triangle the sides are 432, 543, and 654. What are the angles?

Ans. 83° 25′ 13″, 41° 0′ 39″, and 55° 34′ 8″.

Formulas for the areas of triangles.

112. Given two angles and the included side to find the area of the triangle.

Let AC (fig. 33) be the given side, A and C the given angles; employing the same notation as above, we have

$$c = \frac{b \sin \mathrm{C}}{\sin \mathrm{B}} = \frac{b \sin \mathrm{C}}{\sin (\mathrm{A} + \mathrm{C})}$$

the sine of an angle being the same with that of its supplement. Calling d the perpendicular let fall upon the given side, we have $d = c \sin \mathrm{A}$, whence

$$d = \frac{b \sin \mathrm{A} \sin \mathrm{C}}{\sin (\mathrm{A} + \mathrm{C})}$$

multiplying this expression by $\frac{1}{2} b$ in order to obtain the area sought, and denoting this area by A we have

$$\mathbf{A} = \frac{b^2 \sin \mathrm{A} \sin \mathrm{C}}{2 \sin (\mathrm{A} + \mathrm{C})}$$

From which we derive the following rule; *Multiply together the square of the given side and the sines of the given angles; and divide the product by twice the sine of the sum of the given angles.*

Ex. 1. What is the area of a triangle in which one of the sides is 120 feet, and the adjacent angles 40° and 60° respectively?

$$\begin{aligned} \log b^2 = 2 \log 120 &= 4.158362 \\ \sin 40° &= 9.808067 \\ \sin 60° &= 9.937531 \\ \log 2 \text{ Comp} &= 9.698970 \\ \sin 100° \text{ Comp} &= \underline{0.006649} \\ & 33.609579 \end{aligned}$$

or subtracting 30 from the characteristic, on account of complements and tens added,

$$\log 3.609579 = 4068.5 \text{ square feet.} \quad \text{Ans.}$$

Ex. 2. What is the area of a triangle in which one of the sides is 320 yards, and the adjacent angles 30° and 40° respectively?

2. Given two sides and the included angle to find the area of the triangle.

Let a and b be the given sides (fig. 33), C the given angle, we have $d = a \sin C$; whence multiplying the perpendicular by $\frac{1}{2} b$ we have

$$S = \tfrac{1}{2} ab \sin C$$

From which we derive the following rule; *Multiply one-half the product of the two sides by the sine of the included angle.*

Ex. 1. What is the area of a triangle in which two of the sides are 30 and 40, and the included angle 28° 57′?

$$\begin{aligned} \log 2 \text{ Comp} &= 9.698970 \\ \log 30 &= 1.477121 \\ \log 40 &= 1.602060 \\ \sin 28° 57' &= \underline{9.684887} \\ & 22.463038 \end{aligned}$$

Deducting 20 from the characteristic we have

$$\text{Log } 2.463038 = 290.4276 \quad \text{Ans.}$$

Ex. 2. What is the area of a triangle in which two of the sides are 45 and 32 feet, and the included angle 46° 30′?

Ans. 522.266 feet.

3. Given the three sides to find the area of the triangle.

In the triangle ABC (fig. 33) we have $DC = a \cos C$; designating BD by d as before, we have by the known property of a right angled triangle

$$d^2 = a^2 - a^2 \cos^2 C$$

Substituting for cos C its value in art. 110 we have

$$d^2 = a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2$$

whence $$4b^2 d^2 = 4a^2 b^2 - (a^2 + b^2 - c^2)^2;$$

but the two terms of the right hand member being the difference between two squares, we have

$$\begin{aligned} 4b^2 d^2 &= (2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2) \\ &= ((a+b)^2 - c^2)(c^2 - (a-b)^2); \end{aligned}$$

the two factors of this last being each also the difference between two squares, we have

$$4b^2 d^2 = (a + b + c)(a + b - c)(c + a - b)(c + b - a)$$

adding and subtracting c, b and a in the second, third and last factor respectively, and designating $a + b + c$ by $2s$, we have

$$\begin{aligned} 4b^2 d^2 &= 2s(2s - 2c)(2s - 2b)(2s - 2a) \\ &= 16s(s - c)(s - b)(s - a); \end{aligned}$$

dividing both sides by 16 and extracting the square root

$$\frac{bd}{2} = (s(s - c)(s - b)(s - a))^{\frac{1}{2}}$$

Hence, to find the area of a triangle by means of the three sides, *subtract successively each side from the half sum; multiply together the half sum and the three remainders; and take the square root of the product.*

Ex. 1. Given the three sides 49, 50.25, and 25.69 feet, to find the area.

$$\begin{array}{lr} \text{log half sum, } 62.47 & = 1.795672 \\ \text{" 1st remainder, } 13.47 & = 1.129368 \\ \text{" 2d \quad " \quad } 12.22 & = 1.087071 \\ \text{" 3d \quad " \quad } 36.78 & = \underline{1.565612} \\ & 2\,)\,\underline{5.577723} \\ \text{log } 615.75 \text{ square feet} & = 2.788861 \end{array} \quad \text{Ans.}$$

Ex. 2. What is the area of a triangle whose sides are 125, 173, and 216 feet?

Ans. 10809 square feet.

113. We close the subject of trigonometrical analysis with the following problems, for the solution of which we are now prepared.

1. From a window A (fig. 51) in the side of a tower exactly 50 feet from the ground, the angle of depression of the bottom of a may-pole was found to be m°, it was also observed that the angle of elevation of the top of the pole at this station was equal to the complement of its angle of elevation at the bottom of the tower; to determine from these data the height of the tower.

By the question the side AB and the angle BAD are given, and the angle EAC is the complement of ABC; the angle ACD is equal to 90° — ABC, and the angle ACB to 90° — 2 ABC.

In the triangle ACB we have sin ACB : AB : : sin ABC : AC, and in the triangle ACD sin ACD : AD : : sin ADC : AC whence

$$\frac{AD \sin ADC}{\sin(90^\circ - ABC)} = \frac{AB \sin ABC}{\sin(90^\circ - 2\,ABC)}$$

or

$$\frac{AD \sin ADC}{\cos ABC} = \frac{AB \sin ABC}{\cos 2\,ABC}$$

from which we have art. 37, 48, $\tan 2\,ABC = \frac{2\,AD \sin ADC}{AB}$

AD being found by means of the triangle ABD, we have by the above formula the angle ABC, and this being found the height sought will readily be determined.

2. The distances between three points A, B, C (fig. 48) being known, it is required to find the position of the point D with reference to these points, the angles ABD, ADC being found by observation equal to m° and n° respectively.

Let the given distances AB, AC, BC be represented by the letters a, b, c, respectively; in the triangle ABD let the angle ABD be represented by x, and in the triangle ADC the angle ACD by y; the triangles ADB, ADC give the equation.

$$a \sin n \sin x = b \sin m \sin y;$$

but in the quadrilateral ABDC we have

$$y = 4 \text{ right angles} - \text{ADB} - \text{ADC} - \text{BAC} - x.$$

and since the angle BAC is known when the three sides of the triangle ABC are given, the first four terms of the right hand member of this last equation are known; representing these by d for the sake of conciseness, and substituting for y its value in the first equation, we have

$$a \sin n \sin x = b \sin m \sin (d - x)$$

or $$a \sin n\, x = b \sin m (\sin d \cos x - \sin n \cos d)$$

Dividing both sides by $\sin x$ we obtain

$$a \sin n = b \sin m (\sin d \cot x - \cos d);$$

whence $$\cot x = \frac{a \sin n + b \sin m \cos d}{b \sin m \sin d}$$

This problem enables us to determine the position of a point on a plane by means of the angles comprehended between three straight lines drawn from this point to three given points.

BOOK II. SURVEYING.

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SECTION I.

DEFINITIONS. DESCRIPTION OF INSTRUMENTS.

114. The objects comprised in surveying are

1°. The performing such operations as are necessary, in order to a correct representation on paper of certain portions of the earth's surface.

2°. The delineation on paper of such portions from the data obtained.

3°. The computation of their areas or contents.

4°. Conversely, the laying out or marking certain lines or portions of the earth's surface from previous surveys, or from given data.

When the portion of the earth's surface is small, the curvature of the earth may be neglected, and the portion surveyed may, without sensible error, be regarded as a plane surface. The work is then called *Plane Surveying.*

The operations in the field or portion of the earth's surface to be surveyed, consist chiefly in the measurement of certain lines and angles. The instruments usually employed for this purpose are the following.

THE CHAIN.

115. The instrument in general use for the measurement of lines in Plane Surveying is Gunter's chain, so called from the name of the inventor. This chain is four rods in length and is divided into one hundred links. For greater convenience in counting the links, tallies are placed at every ten links of the chain. The length of the chain, moreover, being four rods or 66 feet, a link is a hundredth part of this, or 7.92 inches.

In uneven ground a chain of four rods in length is found inconvenient, and a chain of two rods divided into fifty links is commonly employed. To determine the number of rods, and the decimal parts of a rod contained in a given number of chains and links of this description, we multiply, as it will be easy to see, the chains by two, and the links by four hundredths. Thus in 7 chains 13 links there are 14.52 rods.

116. To measure a line with a chain two persons are necessary, one at each end of the chain. The one that carries the forward end of the chain is called the *leader*, the other is called the *follower*. At the outset the leader is usually provided with ten pins or arrows of about a foot in length, as marks to be placed in the ground at the end of each chain, as the measurement proceeds. He is also provided with some suitable contrivance, as a string attached to his person in which a knot may be tied, to serve as a tally at every ten chains.

In order to measure the distance in a straight line between two points A and B, the following process is observed. A stake or signal being placed at the point B, the follower places his end of the chain upon the point A, and directing the leader right or left as the case may require, the chain is drawn out in the direction from A to the signal at B. When it is near being stretched the follower cries *down*, and the leader, drawing the chain tight, puts down a pin. Both then advance with the chain until the follower again cries *down*. The follower then places his end of the chain at the pin left by the leader, and the chain is again carefully stretched in the direction of the signal at B, and the leader puts down another pin. The process is thus continued, if the line exceeds ten chains in length, until, the chain being still stretched in the direction of the signal at B, and the leader, having put down the last pin, cries *tally*. At this the follower, the chain being still left upon the ground, advances to the leader and takes up the pin last put down, marking carefully the point upon the ground at which it was placed. He then ties a knot in his string as a tally, and passes the ten pins, now in his hands, into the hands of the leader; and the operation is thus continued until the whole line is measured.

In the measurement of a line the chain must be kept *horizontal*. This will require, if the ground is ascending, that the follower raise his end of the chain until it is precisely on a level with the forward end, taking care, when the marking pin is put down, that his end of the chain is in a plumb line passing through the starting point, or the pin last put down. If the ground is descending, the leader must lift his end of the chain in like manner, observing to put down his pin directly under it, at the point where a plumb line passing through his end of the chain would intersect the ground. In either case, if necessary, the chain should be supported at the middle, so as to be kept perfectly straight.

The chief points to be attended to in chaining are 1st, to keep the chain without deviation in the direction of the line to be measured. 2d, to keep the chain in a horizontal position. 3d, to record correctly the number of chains. The process above explained is well adapted to fulfil these conditions.

117. In the process of chaining the following results were found.

1. The follower, at the end of the process, finds five knots tied upon his string, seven pins in his hand, and that there are 15 links between the point where the last pin was put down and the end of the line measured. What was the length of the line? Ans. 114.6. rods.

2. The knots or tallies being 12, the odd chains 5, and the links $27\frac{1}{2}$, what was the distance measured?

Ans. 251.1 rods.

THE SURVEYOR'S COMPASS.

118. The instrument commonly employed in Plane Surveying for the measurement of angles is the Surveyor's Compass. The principal parts of this instrument are a compass box, a magnetic needle, a pair of sight vanes, and a stand for its support.

Upon the plate of the compass box the principal points of the compass are marked by the letters N, S, E, and W. A Fleur de luce is placed at the north point to mark this point more distinctly.

The circular ring of the compass box is divided into degrees and half degrees, each way to 90° from the north and south points respectively.

The sight vanes are constructed with a slit in one corresponding to a thread in the other. They are so adjusted that a plane perpendicular to the face of the instrument and passing through a slit and its corresponding thread, will intersect the north and south line upon the compass plate.

The needle is nicely balanced upon a pivot at the centre of the compass box, upon which, when the instrument is level, it moves with perfect freedom. When left to itself, after a series of vibrations growing shorter and shorter upon either side, it comes finally to a state of rest in a line either due North and South, or in a line at an angular distance from this, capable of being determined for each place where the instrument is used.

The direction in which the needle settles is called the *Magnetic Meridian*, and the angle by which this varies from the true meridian is called the *variation*.

Spirit levels are usually placed upon the instrument to aid in levelling it. Instead of the sight vanes above described, a telescope with cross threads in the axis, is a more perfect arrangement for the sights of the instrument.

The needle, when left to vibrate freely, will, unless there is some local disturbance, come to a state of rest in lines parallel to each other; and hence the capacity of the instrument for the general purpose of surveying.

119. The angular distance between the direction in which the needle settles and the direction of a given line is called the *bearing* of the line or its *course*.

If the direction of the line from the point from which we start, or from which its direction should be estimated, is between the North and East, the bearing is said to be North so many degrees East; if between the North and West, it is said to be North so many degrees West. If the direction of the line is between the South and East, the bearing of the line is said to be South so many degrees East; if between the South and West, South so many degrees West.

120. Let it now be proposed to find the bearing of a line A B (fig. 53) by the compass.

Supposing the point of departure to be the extremity A of the line, for example, and the general direction of the line to be between the North and East, we place the centre of the instrument directly over the point A, and bring it to a level, so that the needle shall have a free and easy motion upon its pivot. The north point on the instrument being from us, we then turn the sight vanes until looking through the slit in the one nearest us, the thread in the one opposite is made exactly to intersect a staff placed vertically at the extremity B, or some other point in the line. The north point on the compass plate will then be on the line, and the distance between this point and that against which the end of the needle settles, estimated in degrees upon the graduated limb, will be the bearing sought.

Ex. 1. The instrument being placed as above, I observed that, when the needle came to a state of rest, it pointed to 40° 30′ on the graduated limb of the compass box. The north point on the compass box being turned from the North toward the East what is the bearing of the line?

Ans. N. 40° 30′ E.

Ex. 2. The north point on the instrument being toward us, suppose that it points to 30° 20′ on the graduated limb, the general direction of the line being between the South and West, what is the bearing of the line? Ans. S. 30° 20′ W.

If the needle has not come to a state of rest after a series of free and easy vibrations, it should again be put in vibration by a gentle tap upon the instrument.

If after taking the bearing of the line AB from the end A, as above, we again take its bearing back from the other end B, the result is called the *back* or *reverse bearing.*

The foward and reverse bearing, it is evident are equal angles. The only difference is that they lie between directly opposite points. Thus, if the bearing of AB from A (fig. 53) is N. 40° 30′ E. the reverse bearing from B will be S. 40° 30′ W.

THE CROSS.

121. The cross is another instrument used in surveying. It consists of two sets of sights, similar to those of the compass, placed at right angles to each other; and a stand for its support.

The use of this instrument is to determine perpendiculars to a given line, either from points upon or without it.

Thus if it be required to determine a perpendicular from a given point upon a line, we place the centre of the instrument directly over the given point, and turning one pair of the sights in the direction of the line, the other sights will mark the direction of the perpendicular to the given point. If the given point is without the line, we place the centre of the instrument upon the line as near as we can judge over the point upon which the perpendicular from the given point would fall, and turn one pair of the sights in the direction of the line. The instrument is then moved to the right or left upon the line, as the case may require, until the other sights are made to intersect the given point. We shall then have the direction of the perpendicular required.

THE THEODOLITE.

The compass is a rude instrument and subject to many inaccuracies. In extended surveys, and when great accuracy is required, the only proper instrument for the measurement of angles is the *Theodolite.* A brief description of this instrument has already been given (art. 71).

122. The instruments described above are sufficient for the operations to be performed in the field or territory to be surveyed. To form a map or plot of the survey, the diagonal scale of equal parts, and the line of chords, art. 10 and 13, will be sufficient. There are other convenient instruments adapted to this object, a few of which we will here explain.

THE SEMI-CIRCULAR PROTRACTOR.

123. The nature of this instrument will be obvious. It is simply a semi-circle graduated into degrees and half degrees, and numbered both ways from 0 to 180°.

Let it be required to lay off from the point A in the line NS (fig. 53) an angle of 40°, for example. We place the edge of the instrument corresponding to the diameter of the semi-circle upon the line NS, and bring the notch, which marks the centre of the semi-circle, upon the point A. Counting the degrees from 0 to 40° on the limb of the instrument, we mark this point upon the paper by a fine prick or dot. Drawing next the line AB through the point A and the prick or dot thus determined, we have the angle sought.

This instrument, it is obvious, is much more convenient than the line of chords for the laying off or measuring of angles upon paper.

THE PARALLEL RULER.

124. This instrument consists of two flat rulers connected by two cross bars parallel and equal to each other. As the instrument is opened, the rulers, will, it is evident, move parallel to each other, and it will thus be found convenient for drawing parallel lines.

Ex. Through a given point B (fig, 53) let it be required to draw a line parallel to the given line NS. We place an edge of one of the rulers upon the given line NS; holding it firmly in this position, we move the other ruler, until one of its edges comes to touch the given point B. Along this edge we then draw a line. This line will pass, it is evident, through the given point and will be parallel to the given line.

THE SECTOR.

125. This instrument consists of two equal arms movable about a pivot as a centre, like a common jointed rule.

We now refer to two only of the lines upon it. These are drawn diagonally from the centre and are divided each into the same equal parts. Let C be the centre (fig. 54) CD, CE the diagonal lines. Let it now be required to adapt the sector to laying down lines upon a scale of 10 feet to the inch. With an extent equal to one inch in the dividers we place one foot at 10 on one of the arms, and open the sector until the other foot of the dividers just reaches 10 on the other arm. Then from the nature of similar triangles, the extent from 5 on one of the arms across to the same figure on the other will be 5 feet on a scale of 10 feet to the inch, and so on for any other figure.

In addition to the above a *square* for raising perpendiculars will be found very useful.

FIELD SURVEYING.

126. We are now prepared with the instruments necessary to measure the bounding lines and the angles required, in order to form a plan and calculate the areas of fields and other like portions of the earth's surface to be surveyed. We commence with the use of the

CHAIN AND COMPASS.

127. Let a field be of the form represented by ABCD (fig. 55), the upper side of the figure being regarded as North, the right hand East, and so on.

Beginning at some convenient point of the field, A for example, we take the bearing of the side AB with the compass, and then measure its length with the chain. Removing next the compass to B, we take the bearing of the side BC, and measure its length as before. And so on, until, passing round the field we return to the point A from which we started. The several bearings and distances may be noted in a book, called the *Field Book*, in the following manner. In the first column we note the number of the station, in the second the bearings, in the third the distances. Thus, in the field under consideration, the minutes of the survey are preserved as follows

Stations.	Bearings.	Distances.	Remarks.
1	N. 23° E.	17 chains.	A Tower bears from 1
2	N. 88° E.	11 "	N. 45° E.
3	S. 14° E.	23 "	From 4 the tower bears
4	N. 77° W.	23.66 "	N. 50° W.

The column headed remarks is designed for entering any thing deserving special notice in the course of the work. In the present example the bearings of a Tower in the field are noted at two stations 1 and 2.

PLOTTING A SURVEY.

128. The necessary operations having been performed in the field, the next work of the surveyor will be to form from his minutes a map or plot of the field surveyed.

To form a plot of the present example, we assume (fig. 55) a point A upon the paper as the first station point, and through this point draw a dotted line NS to represent a meridian line. We then lay off on the right of the line NS an angle NAB equal to the given course 23°. Then through this angle we draw the line AB equal to the distance 17 chains. By aid of the parallel ruler, we draw next, through the point B, another dotted line NS to represent the meridian through B; we then lay off to the right the angle NBC equal to the next course 88°, and through this angle draw BC equal to the next distance 11 chains, and so on for the remaining sides of the field.

If the measurements are all accurately made in the field, and the work is also correctly plotted, on laying off the last line on the required course and of the given length, the end of this line will fall exactly upon the point A, at which we began.

When this is not the case, some error must have occurred, either in the survey or the plotting. If not found in the latter, it should be sought in the field. Such lines and angles being first re-measured, in which it is most probable the error has occurred.

The reverse bearing may be taken, as we proceed, as a test of the accuracy of the direct bearing.

In the present example it remains only to mark on the plot the place of the tower referred to in the remarks. In order to this, at A we lay off to the right an angle equal to the bearing, 45°, of the tower at that station, and through this angle draw an indefinite line. We next at the station D lay off to the left an angle equal to the bearing 50° of the tower from that station, and through this angle draw an indefinite line intersecting the former. The point of intersection of the two lines will be the place of the tower on the plot.

Ex. 1. To plot a field from the following field notes

Stations.	Bearings.	Distances.
1	N. 25° E.	29 chains.
2	N. 70° 30′ E.	40 "
3	S. 26° W.	35 "
4	S. 34° 45′ E.	32 "
5	S. 54° W.	55 "
6	N. 8° 30′ W.	52 "

Ex. 2. To plot a survey from the following notes.

Stations.	Bearings.	Distances.
1	N. 65° E.	40 chains.
2	S. 42° E.	36 "
3	S. 30° W.	39 "
4	S. 25° 30′ E.	49 "
5	N. 70° W.	72 "
6	N. 28° E.	51 "
7	N. 45° 30′ W.	27 "

In the margin of this field book the following notes were made.

From station 1 to 2 the field lies upon a road, which runs in the direction of this side of the field.

From 2 the spire of a church bears N. 68° E; and from 3 the same spire bears N. 15° E.

From 2 also a tree on the westerly side of another road, intersecting the former by the church, bears S. 72° E.; and from 3 the same tree bears N. 70° E.

From 3 another tree by the same side of the road bears S. 41° E; and the same tree from 4 bears N. 70° E.

The corner 5 is on the road. From this corner the road runs by a stone mill which bears S. 44° W. At 6 the mill bears S. 25° E. This corner touches a small stream which turns the mill.

Let the learner from these data trace the roads &c. in connection with the plot of the field.

129. The angles which the sides of a field make with each other may be found from the bearings; and by means of these the field may be plotted without the necessity of drawing meridian lines through the corners.

The angles may be found by the following rules.

1°. If the bearing of one of the sides is north and the other south, one east and the other west, *subtract the less course from the greater.*

2°. If one is north and the other south, and both east or west, *add both courses together.*

3°. If both are north or south, and one east and the other west, *subtract their sum from* 180°.

4°. If both are north or south, and both east or west, *add the less course to the supplement of the greater.*

The reason for these rules may be shown from the example (fig. 55).

Thus to find the angle CDA, we subtract, it is evident, CDN, or which is the same thing, its equal SCD, from NDA.

To find DAB we add SAD, or its equal ADN, to NAB and subtract the sum from 180°.

To find ABC we add NAB, or its equal ABS, to the supplement of NBC.

To find BCD, we add BCS, or its equal NBC, to SCD.

Designating the sides by the number of the stations, and performing the calculations in order from the field book, we have the angle between 1 and $2 = 23 + 180 - 88 = 115°$ $=$ ABC; between 2 and $3 = 88 + 14 = 102° =$ BCD; between 3 and $4 = 77 - 14 = 63° =$ CDA; between 4 and 1 $= 180 - (77 + 23) = 80° =$ DAB.

Ex. Find first the angles and then plot the field from the following notes.

			Ch.
1.	N. 16° 30′	E.	22.
2.	N. 82°	E.	19.60
3.	S. 17°	E.	24.
4.	S. 37°	W.	22.
5.	N. 49°	W.	25.20.

CHAIN AND CROSS.

130. By aid of the chain and cross a field may be surveyed as follows.

Suppose the field to be in the form represented (fig. 56). Commencing at A, for example, we carefully mark by stakes a diagonal line from A to one of the opposite corners, as F. Then by aid of the Cross we determine the points *a*, *m*, *n*, *r*, where the perpendiculars from the remaining corners B, C, D, E will fall upon this diagonal. We next with the chain measure the distances A*a*, A*m*, A*n*, A*r*, to the feet of the perpendiculars B*a*, C*m*, D*n*, E*r*, as also the perpendiculars themselves, and the whole diagonal AF. The field will then be determined.

In order to show the position of the field in relation to the points of the compass, the bearing of the line first measured, or some side of the field, may be taken with the Compass.

The field book may be kept as follows, beginning at the bottom and reading upwards as, in general, is the most convenient method.

	AF	
	18.96	Diag.
E 2.59	13.42	
	11.32	3.25 D.
C 3.47	7.63	
B 7.56	5.22	
Begin	at A	Go E.
Per. on the left	Diag.	Per. on the right.

The method of plotting this field will not require explanation. It is obvious, moreover, that instead of one we may employ two or more diagonals, as shall be found most convenient.

Ex. 1. Let it be required to plot a field from the following field notes.

	CE	
	16.66	Diag.
	13.46	4.96 A.
D 3.76	5.73	
	R Fr.C	
	AC	
	14.33	Diag.
B 2.73	6.43	
Begin	at A	Go W.
	Diag.	

From the field book it appears that two diagonals were measured in the survey. We begin by drawing a line AC (fig. 57) from A towards the left equal to 14.33, the first diagonal. Setting off on this line $Aa = 6.43$, from a we raise toward the left a perpendicular $aB = 2.73$, and join AB, BC; we thus have three corners A, B, and C of the field.

The note R Fr. C indicates that the second diagonal should be laid off from C to the right of the direction AC. In order to construct this diagonal we take in the compasses $4.96 = Ab$, the perpendicular from A on to this diagonal, and, with this extent as radius, from A describe an arc; next with the extent 13.46, the distance from C to the foot of the perpendicular Ab, describe another arc cutting the former in b; then through b we draw the diagonal $CE = 16.66$. Finally we lay off on CE the distance $Cc = 5.73$, and having drawn the perpendicular cD, we join CD, DE, and thus obtain the plot required.

Ex. 2. To plot a field from the following notes.

	CH	
	16.35	Diag.
	9.10	3.90 I.
D 2.35	5.75	
	R Fr. C	
	AC	
	11.65	Diag.
B 2.63	5.30	
	4.00	6.30 I.
Begin	at A	Go S. 67°E.

	EG	
	11.50	Diag.
F 4.00	7.17	
	R Fr. E	
	HE	Diag.
	14.74	
D 4.65	9.75	
	4.65	5.75 G
	L Fr. H	

The above method is eminently adapted to level and open fields, since the lines may be measured with great accuracy, and the work is not liable to errors arising from the imperfections of the compass.

SURVEYS WITH THE CHAIN ONLY.

131. A field may be surveyed with the chain only, by measuring the sides, and a series of diagonals dividing it into triangles.

Ex. 1. Let it be required to plot a field from the following notes.

Sides		Diag.	
EA	5.85	CE	4.20
DE	5.18		
CD	3.65		
BC	3.90	AC	7.68
AB	10.54		

With the sides AB, BC and diagonal AC we construct a triangle ABC, which gives three corners A, B, and C of the field. Next with the diagonal CE and side EA we find another corner E ; and lastly with the sides CD and DE we find the remaining corner D of the field, and the figure is completed.

Ex. 2. To plot a field from the following notes.

Sides		Diag.	
EB	9.30		
CE	6.20		
CD	7.65		
AD	7.50	BC	7.20
AB	16.00	AC	12.88

When the diagonals cannot be measured, we measure the sides of the field, and determine next their inclinations to each other in the following manner.

Let AB, AC (fig. 58) be two sides of the field. The angle BAC being obtuse, we produce BA any convenient distance A*m* ; we then measure on AC another similar distance A*n*, and measure *mn* ; the lines A*m*, A*n*, *nm*, will, it is evident, determine the angle BAC.

Ex. To plot a survey from the following notes.

DA	4.76
CD	6.18
BC	4.43
AB	7.23

1.92 from *m* to *n*.
2.00 from A to *n*, on the side AB.
2.00 from A to *m*, on the side AD.

THE THREE METHODS COMBINED.

132. The three methods above may be combined in the same survey, as circumstances may render necessary or expedient.

A field is sometimes bounded in part by lines of small extent. Instead of taking the bearing of these small lines and measuring them severally, it is, in general, best to run a base, or chain line as it may be designated, in some convenient direction as near as possible to the given boundary, and then determine its principal points by means of offsets.

Suppose from A to B (fig. 59), the field is bounded by a broken line A *b c d e* B. From A to B we measure a chain line AB and take its bearing; the principal points *b*, *c*, *d*, *e*, are then determined by the offsets *mb*, *nc*, *rd*, *se*, in the manner already explained.

Ex. To plot a field from the following notes.

Sta.	Bearing.	Distance.
5	N. 31° W.	9.40
4	S. 70° W.	10.90
3	S. 20° W.	5.30
2	S. 60° E.	11.85 C.L.
1	N. 45° E.	9.30

.00	11.85	C. L. from 2 to 3.
1.50	10.00	
6.00	8.50	
2.00	7.00	
.75	5.20	
2.25	4.00	
1.75	2.20	
.00	.00	

Chain Line.

Instead of a broken line, a field is sometimes bounded by a line irregularly curved, as by the margin of a brook, river, or lake. In this case (fig. 60) we run, as before, a chain line as near the boundary as possible, and by means of offsets determine a sufficient number of points in the curve to draw it to the degree of accuracy required.

Ex. 1. A triangular piece of ground is bounded on two sides by a brook. To plot it from the following notes

AB	N. 37° 30′ E.	3.80	Chain Line.
BC	S. 49° 40′ E.	4.70	“ “
CB	W.	5.85	

	AB
.00	3.80
.30	2.90
.40	2.45
.00	1.90
.00	1.60
.25	.95
.20	.50
.00	.00

Ch. L.

	BC
.00	4.70
.10	4.00
.30	3.60
.50	3.20
.40	2.80
.30	2.20
.40	1.90
.60	1.45
.70	1.00
.45	.50
.00	.00

Ch. L.

In general, if the boundaries of a field are difficult of access or are very irregular, we run chain lines as near them as is convenient, and then determine the prominent points by means of offsets.

Ex. 2. To plot a field from the following notes.

Sta.	Bearing.	Distance.	Remarks.
5	S. 72° W.	5.70 C. L.	From 5 to 1 bounded by a broken line. From 3 to 4 and 5 bounded by a creek. Bounded on this side by a curved road.
4	S. 33° E.	3.00 C. L.	
3	S. 35° W.	3.98 C. L.	
2	N. 71° E.	4.67	
1	N. 15° E.	6.30 C. L.	

.00	6.30
.50	4.60
.40	4.00
.20	3.10
.35	2.10
.60	1.30
.30	.70
.00	.00

1 to 2

.24	3.70	
.00	3.00	
.30	2.20	
.40	1.60	
.25	1.10	
	.60	.22
.00	.00	

3 to 4

.00	3.00
.80	2.30
.90	1.75
.50	1.00
.23	.40

4 to 5

.00	5.70
.25	4.40
.70	3.60
.40	2.80
1.00	2.20
.00	.00

5 to 1

Ex. 3. To plot a tract of land containing several fields from the following notes.

The tract is bounded Northerly by a county road four rods, or one chain, in width; Westerly by a brook running towards the north; and Easterly by a road three rods in width.

	AB	
	14.3	15.65 to C on the county road.
F 2.3	7.9	
	.0	10.05 to G on the county road.
Begin	at A	Go W.

The station B, is on the northerly line of a road, on which several farm houses are situated. The road is three rods in width. Beginning at B, it runs as follows.

	d to G		1.40	
	cd	N. 45° E.	5.30	to the county road.
	bc	N. 75° 30′ E.	2.90	
	ab	N. 29° 20′ E.	4.10	
Begin at B,	B*a*	N. 60° E.	5.04	

The houses are on the north side of the road, at a perpendicular distance of two rods from it. They are parallel to the road, and their position is further determined as follows.

From B, on B*a*, to the centre of the	1st	2.20
" *a*, on *ab*, " "	2d	1.80
" *b*, on *bc*, " "	3d	1.50
" *c*, on *cd*, " "	4th	2.80

Offsets from the perpendiculars AG, BC.

	AG	
	10.05	
	9.00	2.40 D.
	5.60	.90 E.
Begin	at A	

	8.60	2.20
	6.90	1.20
.00	6.50	
.60	4.60	
.90	3.70	
.40	2.25	
.70	1.70	
.00	.00	
Begin	at B	

	BG	
.00	15.65	
.40	14.60	
.80	14.40	
1.40	13.90	
1.20	13.10	
	12.00	.00
	10.90	.80
	9.70	1.40
	9.20	1.30

The Southerly line of the county road passes through the extremities C, and G of the perpendiculars BC, AG.

The tract is divided into fields on the left of the road from B to *d*, by lines parallel to the county road, and running from *a*, *b*, and *c* respectively to the brook; and *a'*, *b'*, *c'*, designating the corresponding points on the right hand side of the road, by lines running from *a'* to F, from *b'* to the middle of FA, and from *c'* to A.

PROOF LINES.

133. If in plotting a field the line last laid down exactly closes the field, it may be inferred, as we have seen, that the operations in the field and the plotting have been correctly performed. In addition to this general test others may be introduced. Thus in fig. 57, we may measure in the field the diagonal DA as a proof line. In the last example the line measured from *d* to G on the side of the road serves the same purpose. For the same object also the line CG may be measured. In general if a field is bounded by a large number of sides, it will be best to establish several proof lines, so that if errors in the operations have occurred, the part of the field in which they were made, may be more readily determined.

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SECTION III.

COMPUTATION OF AREAS.

134. Having seen the method of performing the necessary operations in the field, and of plotting the work from the data thus obtained, we proceed next to the calculation of the areas or contents.

In the measurement of lines in the field the chain has been kept horizontal, and the bearings have been taken in a horizontal plane. The area of the field is, therefore, what it would be, if the whole were projected upon a horizontal plane. The reason for this is 1°, to introduce uniformity in estimating the areas of fields. 2°, Because the surface of an inclined field is no more valuable for building or agricultural purposes, than that of its projection on a horizontal plane. Thus on the lines A*b*, *bc*, (fig. 59) considered as situated on the inclined side of a hill, no more trees or grass can grow, than would grow upon the projections of these lines *am*, *mn*, on a horizontal plane.

1. Let now the field be in the form of a triangle. The area will in this case be equal *to the base multiplied by half the altitude*. It may be found by either of the rules art. 112 according to the parts which are given.

2. If the field is in the form of a rectangle, the area will be equal *to the product of two contiguous sides.*

3. If in the form of a parallelogram, since a parallelogram is equal to twice the triangle of the same base and altitude, the area will be equal *to the product of two contiguous sides by the sine of the included angle.*

4. If in the form of a trapezoid, the area will be equal *to one half the sum of the parallel sides by the altitude of the trapezoid.*

Any figure bounded by right lines may be reduced to some one or more of the above named figures, and thus its area may be computed.

135. Land is usually estimated in acres, roods, and perches, the acre being regarded as the unit of measure. We have, then, according to the table for square measure,

40 square perches = 1 rood
4 roods = 1 acre.

The chain of 4 rods being the linear unit, which is always the case, unless it is otherwise expressed, we have

10,000 square links = 1 square chain,
1 square chain = 16 square perches.
10 square chains = 1 acre.

Ex. Let it be required to find the number of acres, roods and perches in 1878203 square links. The operation will be as follows

$$\begin{array}{r} 18.78203 \\ 4 \\ \hline 3.12812 \\ 40 \\ \hline 5.12480 \end{array}$$

To find the acres we divide the number of square links by 100 000, or which is the same thing, cut off five places to the right for decimals. This gives 18 for the acres. We next multiply the decimal part of the result by 4, cutting off also in the product five decimal places as before, which gives 3 for the rods. Multiplying again, in like manner, by 40 we obtain 5 for the perches. We thus find in the proposed 18 A. 3R. 5. P.

If the area is expressed in chains, and decimal parts of a chain or links, we divide by 10, or, which is the same thing, cut off one decimal place for acres, and proceed as before. Thus in 36.8679 square chains, we have 3 A, 2 R. 29.9. P.

136. Let it now be required to find the area of a field from the following notes.

1.	N. 18°	E.	5.40
2.	N. 41°	E.	7.62
3.	N. 68°	E.	7.50
4.	S. 21°	W.	8.50
5.	S. 6° 20′	E.	9.60
6.	N. 72°	W.	12.25

Having plotted the field, as in fig. 61, we draw diagonals dividing the field into the triangles DCE, BCE, ABE, AEF.

Commenceing next with the triangle DCE, we have, from the measurements in the field, the sides DC, DE, and the angle CDE contained between them; whence, putting A for the area, we have

$$2\,A = DC \times DE \times \sin CDE.$$

We next calculate in the triangle CDE, the remaining angles DCE, CED, and the third side CE; then in the triangle BCE, we have the sides BC, CE, and, subtracting the computed angle DCE from BCD, given by measurement, we have also the angle BCE contained between these sides. Whence in the triangle BCE we have $2A = BC \times CE \times \sin BCE.$

We proceed in like manner to find for the triangle ABE

$$2A = AB \times BE \times \sin ABE.$$

Lastly in the triangle AFE, we have

$$2A = AF \times FE \sin AFE.$$

Adding these several areas together, and taking one half the sum, we shall have the area of the field.

Performing the calculations, which now present no difficulty, we have for the area sought, 12 A. 1 R. 5 P. Ans.

THE RECTANGULAR METHOD.

137. The area of a field may sometimes be found with convenience by a process called the rectangular method which we will now explain.

Let ABCD (fig. 62) be a plot of the field from the field notes.

Through the corners A, B, C, &c. of the field we draw meridian lines, and from each of these corners we let fall perpendicular lines upon the meridian NS which passes through A, the most westerly corner of the field. The perpendiculars, it is evident, will be *east* and *west* lines. The meridian which passes through A is called the first or *prime meridian.*

The perpendicular distance between the east and west lines drawn through the extremities of a course, is called the *northing* or *southing* of the course, according as its direction is north or south.

The northing and southing correspond to difference of latitude, which is to be regarded as north or south, according as it is marked with the signs + or — .

The perpendicular distance between the meridians passing through the extremities of a course is called the *easting* or *westing* of the course, according as its direction is east or west.

The perpendicular distance of a point from the first meridian is called its *departure.*

This being premised, in the triangle A *b* B, with the given course and distance, we calculate the northing A*b*, and the easting B*b*. Next in the triangle B*d*C, we calculate the northing B*d*, and the easting *d*C, and so on for the remaining sides of the field.

The distance C*c* is the departure of the point C, or its distance from the first meridian, and is found by adding its easting *d*C to the preceding easting B*b* of the point B.

The departure *m*D is foumd by adding *n*D the easting of D to the preceding departure *mn* ; and the departure *r*E by subtracting the westing *o*D of E from the preceding departure *m*D, and so on.

To find next the area of the field, we compute first the area of the triangle A*b*B, for which we have the base A*b* and perpendicular *b*B. We next compute the area of the trapezoid *b*B*c*C, for which we have the two parallel sides *b*B, *c*C, and the perpendicular B*d*. Finally we compute the areas of the trapezoids *c*C*m*D, *r*E*m*D, *r*E*q*F, and the triangle FA*q*, for which, it is easy to see, we have the requisite data.

If the figure is now examined with attention, it will be seen that a part of the areas we have computed are connected with lines running north, and the rest with lines running south. The former are called *North areas*, the latter *South areas*. Thus the trapezoid *b*B*c*C is a north area; and *rm*DE is a south area. It will be seen, moreover, that the north areas lie entirely without the field, and that the south areas comprehend the field and the north areas also. Hence, if *from the sum of the south areas, we subtract the sum of the north areas, the remainder will be the area of the field.*

For the sake of convenience we have taken the most westerly corner of the field for that through which to draw the first meridian. The most easterly corner, however, will answer the purpose as well.

Instead of departure we may call the perpendicular distance of a point from the first meridian its *meridian distance*. In this case departure will be limited to express generally the easting or westing of a course, and should be marked by the sign + if the course is east, and by the sign — if the course is west.

In the present example the data and calculations may be conveniently arranged as in the following table.

No.	Courses.	Dist.	N.	S.	E.	W.	M. D.	D.M.D.	Mult.	N.Areas	S. Areas.
1.	N. 18° E.	5.40	5.14		1.67		1.67	1.67	5.14	.85838	
2.	N. 41° E.	7.62	5.75		5.00		6.67	8.34	5.75	4.85300	
3.	N. 68° E.	7.50	2.81		6.95		13.62	20.29	2.81	5.70149	
4.	S. 21° W.	8.51		7.95		3.05	10.57	24.19	7.95		19.23105
5.	S. 6° 18′ E.	9.60		9.54	1.05		11.62	22.19	9.54		21.16926
6.	N. 72° W.	12.25	3.79			11.65		11.62	3.79	4.40398	
		Sum	17.49	17.49	14.67	14.70				15.81685	40.40031

Difference 24.58346

The column marked **D. M. D.** contains the double meridian distances; by which we understand the sum of each meridian distance added to the preceding. We have, therefore, in this column the sum of the parallel sides of the trapezoids, and the bases of the triangles to be computed. In the next column marked multipliers, the northings and southings are, for the sake of convenience, repeated. The product of each double meridian distance into the corresponding multiplier gives the area, written on the same line, in the area columns. By the process the double area is in each case computed; taking half of the result, we have for the area required,

12 A. 1 R. 7 P.

138. Since in the course of the operations we return to the same point of the field from which we started, the sum of the northings, it is evident, should be equal to the sum of the southings, and the sum of the eastings to the sum of the westings. When this is not the case, some error must have occurred either in the operations in the field, or in the computations. And the work must be re-examined, beginning with the most probable source of error, until it is detected.

If the error, however, is trivial, it may be regarded as the result of unavoidable imperfections in the survey and in the calculations. In this case it should be averaged upon all the sides by the following proportion,

As the perimeter of the field,
Is to the length of one of the sides,
So is the error in latitude or departure,
To the correction corresponding to that side.

This correction must be added when applied to a column in which the sum of the numbers is too small, and subtracted in the case where the sum of the numbers is too great.

In the present example, there is a difference of 3 links between the sum of the eastings and that of the westings. By the rule just stated we find the corrections for the sides and the corrected eastings and westings as follows.

	Correction.	Cor. E.	Cor. W.
1.	.003	1.673	
2.	.005	5.005	
3.	.004	6.954	
4.	.005	1.056	
5.	.005		3.045
6.	.007		11.643
		14.688	14.688

Additional columns may be ruled in the table for the corrected latitudes and departures. The corrections should be made and the columns balanced, before proceeding to the computation of the areas.

139. In the example we have considered, the north areas all lie without the field, and do not, therefore, comprehend any part of it. It will sometimes happen, however, in the case of a *re-entering* angle, or one which returns into the field, that a north area will include a part of the field. But in this case the portion thus included will enter twice into the south areas; and the rule for finding the area of the field will still be true.

140. In the calculations above we have found the latitudes and departures by aid of the logarithmic sines, cosines &c. In general, especially if the number of figures in the distances does not exceed three or four, it will be best to use the natural sines.

Radius being unity the natural cosines and sines are decimals, and may be regarded as the latitudes and departures for the distance unity. To find, then, the latitude and departure for a given course and distance, we multiply for the latitude the cosine of the course by the distance, and for the departure the sine of the course by the distance.

To find, for example, the latitude and departure when the course is 32°, and the distance 9, we have

$$\begin{array}{rr} \text{N. cos } 32^\circ = .84805 & \text{N. sin } 32^\circ = .52992 \\ 9 & 9 \\ \hline 7.63245 & 4.76928 \end{array}$$

which gives 7.63 for the latitude, and 4.77 for the departure, true to the second decimal place or nearest link, which is sufficient. In general to attain this degree of accuracy, it will be sufficient to take out the natural sines to the third or fourth place, observing to increase the right hand figure of the part employed by 1, when the left hand figure of the part omitted exceeds 5. Thus, in the example above, we may take .53 for the natural sine, which being multiplied by 9 gives 4.77 as before.

141. The method to be pursued will require no further explanation. The following examples will serve as an exersise for the learner.

Ex. 1. To find the contents of a field from the following field notes.

1.	N. 20° E.	8.50 ch.
2.	S. 76° E.	10.00.
3.	S, 25° W.	13.50.
4.	N. 67° W.	6.00.
5.	N. 18° W.	4.55.

Ans. 11 A. 1 R. 20 P.

Ex. 2. To find the contents of a field from the following field notes.

1.	N. 15° E.	8 ch.
2.	N. 55° E.	6.
3.	S. 20° E.	7.
4.	S. 50° W.	4.
5.	S. 25° E.	8.
6.	N. 61½° W.	11.

Ans. 8 A. 1 R. 8 P.

Ex. 3. To find the contents of a field from the following field notes.

1.	N.	19°	E.	8.40 ch.
2.	S.	76°	E.	5.90.
3.	S.	17°	E.	8.85.
4.	N.	61°	W.	4.60.
5.	S.	24° 30′	W.	6.15.
6.	N.	41° 20′	W.	6.88.

Ans. 7 A. 3 R. 29 P.

142. To facilitate the calculations of latitude and departure, tables have been prepared containing the latitude and departure for every distance from 0 to 100, and for every course in degrees and quarters of degrees from 0 to 90°. Tables of this description are called *Traverse Tables.* They are usually calculated to two places of decimals only ; and as there are various sources of error in the use of them, it is better that the surveyor should make his own computations entire.

THEOREM FOR THE COMPUTATION OF POLYGONAL AREAS.

143. The area of a field may be computed with less labor, than by the preceding method, by the theorem for the computation of the area of any polygon, a demonstration of which, with numerous applications to purposes of surveying is given by Prof. Whitlock of the Genessee Wesleyan Seminary in his Geometry recently published.

The theorem is as follows, *The double area of any polygon is equal to the sum of the products, or combinations of its sides, save one, multiplied, two and two, into the sines of the angles formed by the sides belonging to the several products.*

In the application of this theorem the angles must all be measured in the same direction, and regard must be had also to their algebraic signs as explained art. 95.

Since the number of combinations of the sides increases rapidly with the number of sides, it is best, when the sides exceed four or five, to divide the work into two parts by a diagonal, the diagonal being made the excepted side. When the number of sides are very numerous, two or more diagonals determined by measurement or calculation, may be employed.

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SECTION IV.

VARIATION OF THE COMPASS. MERIDIAN LINES.

144. The compass is a convenient instrument for running courses in woods, but it is subject to many imperfections, and its use in determining boundaries has been a prolific source of uncertainty in respect to them.

VARIATION OF THE COMPASS.

145. The principal cause of imperfection in the instrument arises from the variation, or deviation of the direction in which the needle settles from a true north and south line. This is different in different places, and varies, moreover, in the same place from year to year. The variation at London, for example, which for some years previous had been increasing easterly, reached, in 1580, 11° 17′ east. From this it began to decrease, and in 1657 it became 0, and the needle pointed to the true north. From this the variation commenced in a westerly direction, and increased in that direction until, in 1815, it reached 24° 27′ 18″ west, when it began to decrease; and the direction of the needle is again approaching more nearly a true north and south line.

The following table will show the variation in 1840 for several places in the United States.

Burlington, Vt.	9° 27′ W.	Buffalo, N. Y.	1° 37′ W.
Boston, Mass.	9° 12′ W.	Cleaveland, O.	0′ 19′ E.
Albany, N. Y.	6° 58′ W.	Detroit, Mich.	1° 56′ E.
New Haven, Ct.	6° 13′ W.	Charleston, S. C.	2° 44′ E.
New York City	5° 34′ W.	Cincinnati, O.	4° 46′ E.
Philadelphia,	4° 8′ W.	Mobile, Ala.	7° 5′ E.
Washington City,	2° 0′ W.	St. Louis, Mo.	8° 37′ E.

In the northern states the variation has increased since 1840, about 5 minutes annually; in the middle states from 3 to 4 minutes. In the western states it has decreased from 3 to 4 minutes, and in the southern about 2 minutes annually.

146. There is also a small daily variation depending upon the temperature, which is greatest in the easterly direction about 7 o'clock in the morning and in the westerly direction about 2 o'clock in the afternoon. During the night the needle is stationary. There is also a small annual variation depending upon the same cause. Neither of these are, however, of sufficient amount to be of any consequence in the common operations of surveying.

MERIDIAN LINES.

147. In order to determine the variation at any place, it is necessary to trace a true meridian with which the direction of the needle may be compared. To trace a true meridian is properly a problem of astronomy, for which the learner is not supposed at present to be prepared. The following, however, in the main may be easily understood, and is the best practical method in use.

The method is by observations upon the pole star (Polaris).

If the pole star were precisely at the pole, or the point where the axis of the earth produced pierces the heavens, a line drawn from any point on the surface of the earth in the direction of the star, would be a true meridian. But the star is at a mean distance of about 1° 30′ from the pole, about which it revolves in 23 hours 56 minutes.

There are two ways in which the direction of the pole may be determined by observations upon the star. 1°. By observing the star at the moment when it is on the meridian either above or below the pole. 2°. By observing it when at its greatest distance east or west of the pole, called its greatest eastern or western *elongation.*

When the star is near the meridian, its apparent motion in an easterly or westerly direction is rapid, and the precise moment when it is on the meridian is, therefore, with difficulty observed. But when the star is near its greatest eastern or western elongation, its apparent motion, east or west, is slow; and when at either of these points it does not perceptibly change its direction for some fifteen or twenty minutes. This furnishes opportunity for careful observation. And the time of greatest eastern or western elongation having been carefully noted, the star in 5 hours and 59′ after, will be on the meridian above or below the pole as the case may be. A line drawn at that time in the direction of the star will be a true meridian.

But these methods both requiring the means of accurately determining the time, which are not usually at command, the following is to be preferred, viz, 1°, observe the star, as before, at its greatest eastern or western elongation and mark its direction ; 2°, calculate the angle which the true meridian makes with this direction, which will furnish the means by which it may be laid off.

For the calculation required we have a right angled spherical triangle, in which the complement of the latitude of the observer is the hypothenuse, the angle at the star is a right angle, and the star's polar distance the side opposite the angle required. To obtain this angle we have the following proportion, Cos. Lat : R : : sin Star's polar dist ; sin of the required angle. The bearing of the star, or its angular distance east or west of the pole, as thus determined, is called its *azimuth.*

148. The elongations of the star which answer the purpose, are those which occur in the night. The times in which they occur, approximately at least, it is convenient to know beforehand. They may be found by means of the following tables, which will be sufficient for this purpose for twenty years to come.

EASTERN ELONGATIONS.

Days	April	May	June	July	Aug.	Sept.
	H. M.	H. M.	H. M.	H. M.	H. M.	H. M.
1	18 18	16 26	14 24	12 20	10 16	8 20
7	17 56	16 03	14 00	11 55	9 53	7 58
19	17 34	15 40	13 35	11 31	9 30	7 36
13	17 12	15 17	13 10	11 07	9 08	7 15
25	16 49	14 53	12 45	10 43	8 45	6 53

WESTERN ELONGATIONS.

Days	Oct.	Nov.	Dec.	Jan.	Feb.	March.
	H. M.	H. M.	H. M.	H. M.	H. M.	H. M.
1	18 18	16 22	14 19	12 02	9 50	8 01
7	17 56	15 59	13 53	11 36	9 26	7 38
13	17 34	15 35	13 27	11 10	9 02	7 16
19	17 12	15 10	13 00	10 44	8 39	6 54
25	16 49	14 45	12 34	10 18	8 16	6 33

The time is *astronomical,* and is reckoned from noon. Thus 18 H. 18 M. is 6 H. 18 M. after midnight, or 18 minutes past six in the morning.

Tables also may be computed giving the azimuth for different latitudes for a succession of years. But it is better for the surveyor, by the proportion above, to calculate the azimuth corresponding to his latitude and time of observation. In order to this, it will be necessary to know the distance of the star from the pole at the time of the observation. This distance is variable and is now decreasing. We give below the polar distance of the star Jan. 1st, for a few years, together with the annual variation.

Year.	Polar distance.	An. Var.
1852	1° 28′ 46″.18	19″.256
1853	1° 28′ 26″.93	19″.248
1854	1° 28′ 7″.68	19″.241
1855	1° 27′ 48″.44	

The annual variation may be reckoned at 19,″ sufficiently near for practical purposes for years to come.

Ex. The latitude of Brunswick (Bowd. Coll.) Me, is 43° 53′, and the eastern elongation of the star was observed Sept. 1st, 1852. What is the azimuth of the star ?

Cos. Lat. 43° 53′	9.857786	
Radius	10.000000	
Sin. Pol. dist. 1° 28′ 33″	8.410859	
Sin. 2° 2′ 52″	8.553073	Ans.

149. It remains only to explain the method of observing the star. This may be done with the theodolite as follows.

At fifteen or twenty minutes before the time of the elongation to be observed, as found in the table above, the instrument should be placed at the point from which the line is to be drawn. It should then be properly levelled and sufficient light directed upon the cross threads of the telescope to render them visible.

This being done, the observer will next turn the telescope until the star appears directly behind the vertical thread, and will continue to follow the star until it has reached the point of greatest elongation. At this point it will appear stationary for a time, and then gradually leave the thread of the telescope in a direction opposite to that in which it has been previously moving. The axis of the telescope will thus be left in the direction of the star at the observed elongation.

To mark this direction, an assistant will next hold a lamp, at some convenient distance, in front of the telescope, moving it right or left as may be required, until it appears to the observer directly behind the vertical thread of the telescope. The point thus intersected will then be carefully marked, and a line drawn from the centre of the instrument to this point will be in the direction required.

To mark the meridian line we now calculate the azimuth of the star for the given time and place. Marking a line, which shall make with the line already determined an angle, to the right or left as the case may require, equal to the azimuth found, we shall have the meridian required.

To throw light upon the cross threads of the telescope, a board about a foot square may be preparred movable up and down upon a staff, with a small shelf at the lower edge on which to place a lamp. The board should be covered with white paper, and a hole perforated at the centre sufficiently large for the star to be seen through the telescope. The board being properly arranged before the telescope, the threads will be sufficiently illuminated, while the star at the same time is seen.

Instead of laying off the azimuth angle by the theodolite, after marking the bearing of the star we may assume any given distance upon this bearing as a base, and calculate with this distance and the azimuth, the perpendicular distance to the true meridian; by means of which the meridian may be determined as before.

In case a theodolite is not at hand the following method of observing the star may be employed.

Let two stakes be driven in the ground about four feet apart in a direction east and west; and on the top of these let a joist be nailed, smooth on the upper surface, which should be made perfectly level. At about twelve feet in front, let a plumb line be suspended from the top of an inclined stake, of sufficient height for the star to be seen behind the line from the horizontal bar. The plumb should be swung in a vessel of water to keep it from vibrating.

Upon the horizontal bar let a sight vane of the compass be placed, and moved to the right or left, as may be required, until the observer, looking through the slit in the vane, sees the star at its greatest elongation directly behind the plumb line. This being done, the direction of the star at its greatest elongation will be known as before.

In this operation it will be neccessary to have the plumb line lighted, which may be done by an assistant holding a lamp near it.

Having established a meridian line, the variation of the compass is easily observed. All we have to do is to place the compass upon the line, and note the angle which the needle makes with it.

150. A meridian line should be established in some convenient place in every town, by means of which the variation of the compass may, at any time, be ascertained. And in all conveyances of real estate, the boundaries of which are determined by the compass, the variation of the compass, at the time the boundaries were run, should be careful noted. The want of such lines, in addition to the carelessness of early surveys with the compass, has given rise, in most parts of the United States, to innumerable suits at law for the determination of questions relative to the boundaries of estates, as well as the expense and ill will between the parties consequent upon them.

151. To avoid these difficulties, the General Government in 1802 adopted the following method of surveying the public

lands, proposed by Colonel J. Mansfield then surveyor-general of the North-Western Territory.

1°. Through or near the middle of the tract surveyed a meridian line is run, and designated by permanent marks. This is called the *principal meridian.* Parallel to and on both sides of this, other meridians are run, at the distance of every six miles.

2°. At or near the middle of the principal meridian and perpendicular to it, an east and west line is run, called the *principal parallel* or *base line.* On both sides of this, at the distance of every six miles, other parallels are also run, dividing the territory into equal squares, six miles on a side, and containing 36 square miles each. These squares are called *townships.*

3°. Each township is again divided into smaller squares called *sections,* by lines run parallel to the meridians and parallels, at the distance of one mile from each other. A section, therefore, contains one square mile or 640 acres. The sections are again divided into quarters, eighths or sixteenths.

The townships which lie along the same meridian are called *ranges.*

The ranges are numbered east or west of the principal meridian. The townships are numbered north or south of the base line. The sections are numbered, beginning at the north east corner of the township, from east to west and west to east progressively, to 36 the last number, which is found in the south east corner of the township.

A particular section is denoted thus, section 10, township 3 north, in range 5 east.

To find the section, therefore, we seek the fifth range of townships east of the principal meridian, and the third township in this range north of the base line will be the township. The tenth section in this township will be the section sought.

New principal meridians are run and properly numbered when necessary. New principal parallels are in like manner also run.

In consequence of the convergency of the meridians, the townships, laid down as above, will not be perfect squares, the north bounding lines being less than the corresponding south ones. To prevent the excess or deficiency in the area of the townships arising from this source from accumulating, new parallels are measured at suitable distances on each side of the base line, called *standard parallels*, or correction lines.

The standard parallels north of the base line serve as bases for the townships north of them, and those south of the base line for the townships south of them.

In the original surveys the townships are indicated by posts set in the ground, square at the top, and properly marked and numbered. When a post is designed to indicate the corner of four townhips, diagonal lines are drawn across the top, and the post is set so that these lines will range, one due north and south, and the other due east and west. When a post marks the corners of but two townships, as will be the case on the standard parallels, it is set so that its faces will range due north and south, and due east and west.

The corners of the sections and quarters of sections are also indicated by posts or other permanent marks.

In running out the sections in a township, the excess or deficiency of area in the township arising from the want of parallelism in the meridians, is usually thrown upon the most westerly tier of sections; and the precise area of each of these is determined and suitably noted. The other sections will then contain each the required number, or 640 acres.

The principal meridians are numbered from east to west, and, with the base lines, are characterised by peculiar marks. Thus, the surveys in Wisconsin are all referred to the fourth principal meridian, which starts from the mouth of the Illinois river. The base line is the southern boundary of the State, so far as it borders on the State of Illinois.

The standard parallels are run at a greater or less distance from each other according to the convergency of the meridians. Thus in the surveys in Oregon, it is proposed to run the standard parallels north of the Columbia river at a distance of four townships or 24 miles from each other, and south of that river at the distance of five townships or 30 miles from each other. In general the distance between the parallels should not exceed sixty miles.

The principal meridians and parallels are determined by astronomical observations. The subdivisions are run by the compass.

SECTION V.

LEVELING.

152. A common operation in surveying is to determine the difference of level between two points on the surface of the earth.

Two points are said to be on a level when they are equally distant from the centre of the earth.

The surface of a fluid at rest is a level surface, since all its points are at equal distances from the centre of the earth. Any two points, therefore, are on a level, when they are equally distant from the surface of a tranquil fluid, supposed to be situated immediately above or below them.

A level surface, moreover, is one that is everywhere perpendicular to a plumb line, or the radius of the earth considered as a sphere. This is called a *true level.*

153. A plane surface perpendicular to the radius of the earth at one point only, is called an *apparent level.*

Thus if a line AB (fig. 63) is drawn perpendicular to the radius AC of the earth at A, any two points A and B of this line are on the same apparent level. And if a line BC be drawn to the centre of the earth intersecting its surface at D, BD will be the difference between the true and the apparent level of the point D as determined from A.

To find this difference we have (Geom. B. IV. 30)

$$AB^2 = BD \times (BD + 2\ DC)$$

But for any distance we have occasion to measure, BD is very small compared with 2 DC, or the diameter of the earth. BD may, therefore, by the side of 2 DC be neglected without sensible error. The arc AD and its tangent AB, moreover will not, for small distances, essentially differ ; we have, therefore

$$AD^2 = BD \times 2\ DC \qquad (1)$$

whence

$$BD = \frac{AD^2}{2\ DC}$$

That is, the difference between the apparent and true level of any two points, is equal *to the square of the distance divided by the diameter of the earth.*

Ex. Let AD = a statute mile, or 5280 feet, and 2 CD, the diameter of the earth, = 7912 miles ; we have

$$BD = \frac{(5280)^2}{7912 \times 5280}$$

or, performing the operations, which is done most conveniently by logarithms, we have

$$BD = 8.0076 \text{ inches.}$$

Thus, the difference between the apparent and true level for a distance of one mile is 8 inches.

If we take any other distance AD′ (fig. 63) we have

$$AD'^2 = B'D' \times 2\ DC\,; \qquad (2)$$

comparing the equations (1) and (2), we obtain

$$BD : B'D' :: AD^2 : AD'^2.$$

Whence, *the difference between the apparent and true level for different distances, is as the square of the distance.*

Ex. What is the difference between the apparent and true level for two points distant $2\frac{1}{2}$ miles from each other.

By the proportion above, we have

$$1^2 : 2\tfrac{1}{2}^2 : : 8 \text{ inch.} : 50 \text{ inch.}$$

To find the difference of level in inches, therefore, it will be sufficient to multiply the square of the distance in miles by 8.

154. We proceed next to explain the method for finding the difference of level between points situated at a moderate distance from each other on the earth's surface.

The instruments employed for this purpose are the level, leveling staves, and chain.

LEVEL.

155. The essential parts of the level are 1°, a telescope with cross threads in the axis like the theodolite, and an attached spirit level, so adjusted that when the bubble is at the centre the axis of the telescope is in a horizontal line. 2°. A tripod with two brass plates, to the lower one of which the legs are attached. The upper plate is connected with the lower by means of a ball and socket joint, and is capable of being moved upon it in all directions by four screws, called the leveling screws. The telescope, when in use, is placed upon the upper plate, having a free motion about a pin at the centre. By means of the leveling screws the bubble may be brought to remain at the centre of the attached level, and thus the axis of the telescope continue in the same horizontal plane, in whatever direction the telescope is turned.

LEVELING STAVES AND CHAIN.

156. The staves employed are usually ten feet in length, divided into feet, and tenths and hundredths of a foot. A tar-

get is made to slide up and down the staff, with a line across it at right angles to the staff, marked by different colors on opposite sides for the sake of distinctness.

The chain commonly employed is 100 feet in length, divided into 100 links. A link is, therefore, one foot.

157. Let it now be proposed to find the difference of level between any two points A and C, (fig. 64).

We place the instrument at some convenient point B between A and C, and bring it to a level. An assistant holding a staff in a vertical position at A, we turn the telescope toward it, directing the target to be slipped up or down, as may be required, until the horizontal thread of the telescope intersects the cross line of the target at m, for example. We shall then have Am the distance of the line of intersection from A, or the distance of A below the horizontal line mn. The assistant then removes the staff to C, and the telescope being turned toward it, we find in like manner the distance Cn of C below the horizontal line mn. The difference between Am and Cn will be the difference of level between the points A and C. Thus suppose $Am = 10.75$ feet, $Cn = 7.30$, the difference of level between A and C is 3.45 feet.

If the points are at a considerable distance from each other as A and G, for example, several intermediate stations as C, E, &c. may be necessary, which are taken at convenience. Having found in this case the difference of level between the points A and C as before, the instrument is moved forward to some convenient position D between C and E, and the difference of level between these points is determined in like manner, and so on to G.

158. In this series of observations the sights in the direction from the instrument toward A, the point of departure, are for the sake of distinction called *back sights*, and those toward G are called *fore sights*. If the ground is ascending

the back sight will be greater than the fore sight, if descending it will be less. In either case the difference between the fore and back sights will be the difference of level between the two consecutive sights, and should be marked + or —, according as the ground is ascending or descending.

The field book may be kept as follows

No.	B. S.	F. S.	Diff.	Total.
1	4.52	1.32	3.20	3.20
2	5.48	2.53	2.95	6.15
3	6.72	4.25	2.47	8.62
4	2.43	7.53	— 5.10	3.52
5	1.20	6.45	— 5.25	— 1.73

In the first column is the number of the station, in the second the back sights, in the third the fore sights. In the next the differences, marked + or —, as the ground is ascending or descending. In the last column is the total or sum of the differences, which shows the difference of level between each successive station and the first; and finally the difference between the points required. In the present example the last point is 1.73 feet below the starting point.

In the operations the level should be placed as nearly as possible at the centre between the stations; since in this case no correction will be necessary for the difference between the apparent and true level.

Ex. 2. Between two points A and B the following levels were taken, what is the difference of level between the two.

No.	B. S.	F. S.
1	3.75	1.20
2	4.32	1.83
3	2.03	5.67
4	1.48	3.84
5	5.26	1.92
6	4.73	2.29
7	1.05	4.63
8	.43	4.20

Ans. B is 2.53 feet lower than A.

159. In the column marked total, we have the differences of level between each successive station and the point from which we start. By means of these and the distances between the stations, which should be measured, a profile of the ground between the extreme stations may be drawn.

Thus, let the distances between the stations be 25, 30, 50, and 40 feet, and the total differences of level corresponding be 10, 7, 12, and -3 feet respectively, to draw the profile.

Upon an indefinite line AB (fig. 65) representing a horizontal line through the first station, we set off A*b*, *bc*, *cd*, *de* equal respectively to the distances 25, 30, 50, and 40; then at the point *b* we erect the perpendicular $bm = 10$, the first difference of level, and from *c*, the perpendicular $cn = 7$, the second difference, and so on. The last perpendicular, being negative, is set off below AB. We next through A and the extremities *m*, *n*, *o*, *p* of the perpendiculars, draw the broken line A*mnop*, which will be the profile required.

The level is employed chiefly in the survey of routes for canals and railroads. We shall explain more fully the course usually pursued in surveys for the latter object.

SURVEYS FOR RAILROADS.

160. The object of the survey is to enable us, 1°, to draw a plan of the route, and to represent upon the plan the leading characteristics of the surface over which the route passes. 2°. To draw a profile of the route.

The best line for the direction of a railroad is the straight line joining its two extremities. This, however, is seldom feasible, except for very short distances. In general we are obliged to follow the courses of streams, to wind round among valleys, &c. in order to avoid high hills and other obstructions.

PLAN OF THE ROUTE.

161. A careful examination of the country between the extremes of the road is made, and the general course of the route selected. This is called a *reconnoissance.*

Planting signals along the route, at every point where a change is made in the course, we take in succession the bearings of the stations, thus marked, from each other, and measure the distances between them. We also take the bearings of such points or objects on either side of the route, as we wish to notice particularly on the plan; or, as may be more convenient, we determine their positions by means of offsets.

The character of the ground, or its *topography* as it is termed, is also sketched at the same time—that is, the general nature of the surface, whether it is covered with wood or is cultivated; whether it is meadow or high land, or rocky, marshy, &c. Particular signs have been agreed upon by which to represent these circumstances to the eye, which will be hereafter explained.

162. The following method is usually employed for keeping the field book.

The book is ruled across with parallel lines, at a convenient distance from each other, to represent a chain, or any number or parts of a chain at pleasure.

The left hand page is ruled, up and down, in convenient spaces to write down the number of the station, the bearing, and distances of the stations, &c.

The right hand page is divided into two parts at the middle by a single vertical line.

Using a **O** to indicate a station point, and beginning at the bottom of the page, we place a **O** at the intersection of the vertical with the bottom line, and against it, on the same line in the left hand page, we enter the number, bearing, and length of the first course. Let the length of this course be 5 chains, for example. We take 5 of the horizontal spaces to represent this distance, and placing a ⊙ on the vertical line of the right hand page, at the fifth line from the bottom one, to indicate the second station point, against this, on the same line on the left hand page, we enter, as before, the number, bearing, and length of the second course ; and so on.

The topography is then sketched, by aid of the proper signs, on either side of each course as we proceed. In like manner the forms and relative position of objects of minor importance may be noted. Or, when accuracy is required, their position as determined by bearings or offsets may be noted.

The distance of an offset from a station on the line of the course is entered in the column of distances on the left hand page. The perpendicular distance is entered on the right hand page, beneath a perpendicular line, drawn from the line of the course to the point or object determined by the offset.

Thus, between stations 1 and 2 the line of the route may pass through cultivated fields; between 2 and 3 through a grove of oaks or other trees; between 3 and 5 along the margin of a brook, with meadow or marshy ground on either side. These should all be properly sketched in the field book. The margin of the brook may be traced sufficiently near for all practical purposes by the eye, by observing carefully its windings in relation to the line of the course near it. Or, if greater accuracy is required, offsets to it may be made to as many points as we please.

The points at which the line crosses roads, creeks, brooks, &c., should be carefully noted and entered in the field book.

In this manner, it is obvious, a very perfect map may be made of the whole route, giving at each point all the information necessary for the purposes of the survey.

PROFILE OF THE ROUTE.

163. Upon the whole line of the route, as thus determined by the chain and compass, pins, beginning at the point of departure, are driven into the ground at the distance of one chain, or 100 feet, from each other, to serve as stations for the levels. They are numbered, beginning at the first, 0, 1, 2, 3, &c. Thus the number of the station will give its distance in chains, or multiplied by 100 its distance in feet, from the first station, or point from which we started.

If intermediate stations are required, as in crossing a ravine, for example, pins are driven into the ground upon the edge of the banks, bottom, &c., and their distance from the primary station next preceding marked upon them.

The stations being thus prepared, the next operation is with the level.

To save labor the instrument should be placed successively in positions to observe, at the same time, as many of the stations as possible, taking care always to place it as near the centre of the extreme stations observed as practicable, in order to avoid the corrections for level.

The observation upon the staff at the 0 station, or point of departure, is the *first* back sight; all others are regarded as fore sights in whatever direction the instrument is turned. The station upon which the last fore sight is taken, becomes the station for the *second* back sight and so on. The differences between each sight and the one next following are the differences of level, which are marked + or —, according as the ground is ascending or descending.

The stations at which the level is moved are called *changing stations.* Observations upon these should be made with great care, as any error upon them will affect the whole subsequent work. Errors between the changing stations balance each other.

164. Let it be proposed next to plot a profile from the following notes.

	No.	B. S.	F. S.	Diff.	Total.
	0				12.00
	1	8.00	4.00	4.00	16.00
	.20		5.25	— 1.25	14.75
	.50		9.75	— 4.50	10.25
	2		2.00	7.75	18.00
	.50		.00	2.00	20.00
Pt.	3		4.00	— 4.00	16.00
	.30	4.30	1.05	3.25	19.25
	.70		13.80	— 12.75	6.50
	4		2.30	11.50	18.00
	5		5.70	— 3.40	14.60
	6		3.55	2.15	16.75
Pt.	Pin		3.30	.25	17.00
	7	1.63	5.13	— 3.50	13.50
	8		9.81	— 4.68	8.82
Pt.	9		15.06	— 5.25	3.57
	10	5.06	6.13	— 1.07	2.50
	11		2.63	3.50	6.00
	12		2.13	.50	6.50

The differences of level are usually small in comparison with the distances between the stations. On this account, it is usual to plot the former on a much larger scale than the latter. The proportion usually employed is 1 to 20. Thus 5 feet in the differences of the levels would correspond to 100 feet between the stations. The profile in this way is greatly exagerated, but it occasions no error, since it is the differences of level only that we wish to know at different points.

To facilitate the plotting of levels, ruled paper is prepared, with parallel lines at a convenient distance from each other to represent a foot. These are intersected by perpendicular lines at a distance from each other equal to five times that of the former. On the scale employed for the distances between the stations, each of these spaces will therefore be 100 feet.

It is usual in carrying out the totals in the field book, to assume the starting point, or 0 station, at a sufficient height above the line to which the levels are referred, to render the totals all positive. We thus avoid the inconvenience of negative signs, or of getting below the ruled paper.

The assumed height, it is evident, is a constant added to all the levels and does not affect their relative value. It is altogether arbitrary, except that it should be taken sufficiently large not to be run out in the course of the work.

The course to be pursued will now be sufficiently evident from inspection of fig. 66, in which the plotting is carried to the 5th station. The differences of level are plotted on a scale of 20 feet to the inch, and the distances between the stations on a scale of 200 feet to the inch. The horizontal lines are drawn at the distance of 2 feet from each other.

As an exercise let the learner prepare the paper and plot the whole work, upon the scale here employed; and also upon the scale of 5 feet for the levels to represent 100 feet between the stations, drawing the horizontal lines at the distance of one foot from each other.

165. Having plotted the work the next thing will be to determine upon it the line or grade of the road. This should be done in such a manner that the cuttings above the grade shall fill the cavities below with the least possible moving of the earth. An experimental line is first drawn on the profile, and then varied, until, by calculation, it is found best to fulfill the required condition. The grade, moreover, may be varied from point to point at pleasure, so long as it is kept under 18 or 20 feet to the mile. In no instance, if possible, should it ever exceed 40 feet to the mile. The profile, to-

gether with the character of the ground, through which the cuttings are to be made, furnish the means of deciding upon the grade. In this, however, there is much room for the exercise of judgment.

166. In the survey of the route thus far, the turns are at angles more or less obtuse. This is inadmissible in the actual construction of the railroad, and the turns must be made on curves uniting the lines which form an angle in such a manner that these lines will both be tangents to them. The radius of the curve which thus unites two lines of different bearings, should in general be as large as practicable, but at least should never be less than 1000 feet.

SECTION VI.

TOPOGRAPHICAL SURVEYING.

167. The object of topographical surveying is to delineate the contour and to describe minutely portions of the earth's surface.

The method for determining the form of the ground, its undulations and inequalities, consists, in general, in drawing on paper a series of curves representing the intersections with the surface of the ground of a series of horizontal planes taken at a given distance from each other. These curves, it is evident, will approach or recede from each other, as the ascent in any part is more or less steep.

Let it be proposed, for example, to determine the contour of a hill having its summit at A (fig. 67.)

Driving a stake at A to mark this point, we mark down the sides of the hill as many lines AB, AC, AD, &c., as we judge necessary for our purpose. We then place the level at

A, and put the target on the staff at a distance equal, in addition to the height of the instrument, to the distance we wish the secant planes to be from each other. The instrument being brought to a level, we next move the staff up or down one of these lines AB, for example, until, at the point *b*, the target is intersected by the horizontal thread of the telescope. Driving a stake at *b*, we next move the staff to the line AC, and determine, in like manner, the point *c* in this line; and so on for the remaining lines AD, &c. The points *b*, *c*, *d*, &c., it is evident, will be on the same level, and the line which joins these will be the line in which the secant plane, at the proposed distance below A, comes out on the sides of the hill.

In like manner, we determine the points *m*, *n*, *o*, &c., at the same distance below *b*, *c*, *d*, &c., that these last are below A. We thus determine the intersections with the surface of as many secant planes, at the same distance from each other greater or less, as we deem sufficient, according as the hill is more or less steep, or as a greater or less degree of accuracy is required.

In this operation the instrument should be kept at A, until as many curves as possible are determined from that station. Let *m*, *n*, *o*, &c., be the points last determined from A; we move the instrument next to some position on, or a little above, the line which joins these points, from which to observe as many points as possible in the next line below. The place of the instrument may have to be changed several times before this next line is determined.

This being done, we next measure the distances A*b*, *bm*, A*c*, *cn*, &c., taking care to keep the chain horizontal, since it is the horizontal distances between these points that we want. We measure also the angles BAC, CAD, &c., or, which will answer the same purpose, the lines *bc*, *cd*, &c.

In the present example let the following be the field notes.

Difference of level between the secant planes 8 feet.

Distances on the line AB, $Ab = 15$ feet
$bm = 21$ "
$mB = 10$ "

Distances on the line AC, $Ac = 14$ "
$cn = 12$ "
$nC = 17$ "

Distances on the line AD, $Ad = 14$ "
$do. = 10$ "
$oD = 18$ "

Angles, $BAC = 30°$, $CAD = 33°$.

168. To plot the work we draw at pleasure a line AB (fig. 67) on which we lay off the distances Ab, bm, mB, $=$ 15, 21, and 10 feet respectively. We next make the angle $BAC = 30°$ through which we draw AC, and on this line lay off the distances Ac, cn, nC, $=$ 14, 12, and 17 feet respectively, and proceed in like manner for the line AD. We then draw through b, c, d, the curve $b\,c\,d$, and through m, n, o, the curve $m\,n\,o$, &c. The curves thus drawn will give an idea of the general form of the hill. Thus between B and m the ascent is very steep; between D and o it is much less so.

The general shape of the hill may also be made more apparent to the eye, by a proper shading of the parts between the curves. This is done by drawing fine lines perpendicular to the horizontal curves.

The difference of level between the planes, and the distances measured, furnish data for drawing profiles of the hill in the direction of the lines AB, AC, &c., by which its form may be still further exhibited.

Ex. To determine the contour of a hill from the following notes.

AB	AC	AD	AE	AF	AG	AH
10	7	6	8	15	5	8
5	6	6	12	12	8	4
22	17	12	10	7	8	11
35	12	10	19	10	9	11
29	16	19	11	9	11	12
27	17	10	8	6	12	5

AB, AC, &c. denote severally the lines drawn from the vertex A down the sides of the hill. The numbers beneath them denote respectively the distances from each other of the points where the secant planes come out upon these lines. The following are the angles contained between the lines.

BAC = 24° FAG = 60°
CAD = 40° GAH = 74°
DAE = 69° HAB = 45°
EAF = 48°

169. We have found the points A, *b*, *m*, &c. so that the differences of level between them shall be equal to the assumed distance between the secant planes. This is the most convenient for plotting, though not the most convenient for the operations in the field. In general, therefore, the following course may be pursued.

Along the line AB drive stakes at *b*, *m*, B, &c. as is found most convenient, or as we judge will best subserve our purpose, and take the differences of level between these points, and measure also the distances A*b*, *bm*, &c. as before. We then regard the differences of level as uniform for the distances A*b*, *bm*, &c., and having assumed a convenient distance for the distance between the secant planes, we determine by a simple proportion the points where these planes will come out on the line AB. We proceed in like manner upon the remaining lines AC, AD, &c. and then plot the work as before.

Suppose, for example, that Ab is 12 feet, the difference of level between A and b is 8 feet, and that we wish the secant planes to be at the distance of 6 feet from each other. The first secant plane will then be 2 feet above the plane passing through b; and to find where on Ab the secant plane cuts the ground, we have

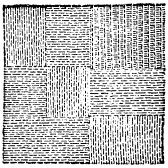
$$8 : 12 :: 2 : 3.$$

The secant plane cuts the ground, therefore, at 3 feet above b, or 9 feet below A. The proportions required in this case are performed practically with great facility by aid of the sector.

170. It remains to explain the manner of noting the character of the surface, whether composed of rocks or covered with wood, or cultivated fields, or as containing portions of meadow, marshy ground, pools of water, &c.

For this purpose certain characters, or conventional signs, have been adopted by the U. S. Topographical Bureau, and are used in all plans and maps made by the U. S. Engineers.

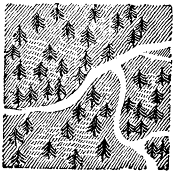
Fig. 1.



Cultivated ground is represented by fine interstitial, or dotted lines, parallel to each other as in figure (1.)

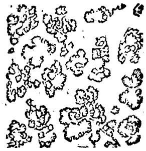
Gardens are represented as cultivated ground, but laid out in a more ornamental manner with walks, trees, &c.

Fig. 2.



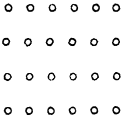
Wood lands are represented by trees drawn as they appear standing; the different forms of the trees indicating the different kinds of wood land. Thus figure (2) represents a pine wood.

Fig. 3.



It is in general, however, most convenient to represent wood lands by trees as they would appear in projection upon the ground, as in figure (3.)

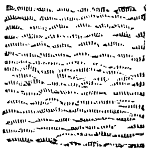
Fig. 4.



Orchards are represented by trees in parallel rows, or by very minute circles arranged in the same manner and indicating the projections of the trees on the ground as in figure (4.)

Grass ground is indicated by clusters of short vertical lines or strokes of the pen, representing tufts of grass, and arranged uniformly over the surface.

Fig. 5.



Pasture is represented by the same kind of clusters scattered unequally over the surface as in figure (5)

Flowing water, as rivers, brooks, &c., is represented by curvilinear or wavy lines drawn parallel to the shore. The lines are drawn fainter and the intervals between them increase as they recede from the shore. A brook is represented in figure (8).

Standing water, as lakes, ponds or pools, is represented by fine horizontal lines drawn from left to right either entirely across the surface, or for a short distance around the shore.

Fresh marsh is represented by pools of water interspersed with clumps of grass as in figure (6).

Salt marsh is represented by horizontal lines with scattered tufts of grass as in figure (7).

Meadows are represented as in figure (8).

Fig. 6.

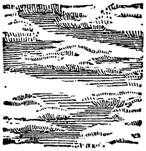


Fig. 7.

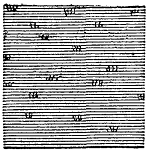
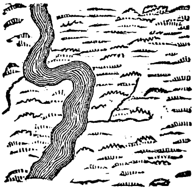


Fig. 8.



Heath is represented by tufts of grass interspersed with bushes and shrubs.

Deep morass is represented in the same manner as fresh marsh, with the addition of a few bushes and clusters of short irregular vertical strokes to represent flags.

Sand is represented by dots. Thus a sandy shore is represented by dotted lines parallel to the line of the shore.

Other signs are employed to denote the different objects which occur, such as mills, churches, &c.

171. We have explained the manner in which the topography in a railroad survey may be sketched in the field book. Let the learner now draw a plan of a route, with its topography, from the following field notes.

Ex. The route commences at the outlet of Long Lake and extends to Broad Bay by the following courses and distances.

Sta.	Bearing.	Dist.
6	N. 63° E.	30.75 chs.
5	N. 32° E.	26.40
4	N. 20° W.	30.00
3	N. 5° E.	20.00
2	N. 25° E.	26.50
1	N. 45° E.	22.30
0	N. 60° E.	25.00

NOTES.

1. From 0 station to 2, the line is on the westerly side of a road, two chains in width, leading from the outlet of the lake to the bay.

2. From 0 to 1 the line is also on the right margin of a brook, which, proceeding from the lake, makes a turn northerly at 1 and is again met by the line at 3. From this station the brook proceeds in a direction S. 80° E. At 2 an offset to the brook was found to be 11 chains, and at 13 chains from 2 an offset is 9 chains.

3. From 2 the road from the lake runs N. 47° E. to its termination at the bay.

4. On the right of the road from 0 to 1 is a thick pine grove; from 1 to 2 is pasture land; and from 2 to the intersection of the road with the brook, is a cultivated field. On the left between the road and the brook is meadow land.

5. From 0 to 3 on the left of the brook is a pine wood. From 3 to 4 the line passes over a fresh marsh. From 4 the ground rises gradually to 5, and is cultivated on both sides of the line.

6. On the left from 5 to 6 is a grove of oaks. At 6 the line meets the right bank of a river which, coming down from the north west, makes a sudden bend at this point. From 6 to its termination at 7 the line is on the bank of the river. At 6 the river is 3 chains in width and at 7 it is 4 chains. Commencing at the point opposite 7, the following courses were run along the line of the river and bay, viz; 1, N. 10° E. 15 chains; 2, N. 24° E. 21 chains.

7. Through 6 passes the westerly line of a road, intersecting the lake road to the bay at right angles. From the point of intersection to the bay is 40 chains. And at 25 chains, toward 6, from the point of intersection, an offset to the bay is 20 chains. The ground on both sides of the cross road is occupied by dwelling houses, gardens, orchards, &c.

8. At station 4 the summit of a high hill was observed bearing N, 70° W. At 5 the bearing of the hill is S. 45° W.

At 5 also the bearing of another hill is N. 40° W. At 6 the bearing of the same hill is N. 87° W.

172. The above notes are designed as general directions merely, and may, with the courses and distances, be varied at pleasure. We close this subject with the following example.

Ex. To trace the passage of a river through the notch of two mountains.

To trace the river, beginning at a station 0 on the west side, we take the following bearings, distances, and offsets to the line of the shore.

Sta.	Bearing.	Dist.	Offsets.	
N.		Chs.	L.	R.
		5.00		.30
		1.40	.15	
M.	N. 30° E.	0.30		.10
		3.00		.40
		2.00	.00	.00
C.	N. 50° E.	0.50		.15
		3.00		.20
		2.20	.15	
		1.30		.20
		.80	.00	.00
B.	N. 20° E.	0.20		.07
		4.40		.20
		2.00		.40
0.	N. 10° E.	0.00	.00	.00

In the column of distances the distances of the offsets from the beginning of the course are written in order from the bottom upward, the number last written being the whole length of the course, or distance. Thus 4.40 is the first distance, 3 the second, and so on.

The breadth of the river at 0 is 1.80; at M 60; and at N 50 chains. From B to station *b*, at the water's edge, on the opposite shore is 1.20 chains; from C to *c* do. 1.05 chains; each on a perpendicular to BC.

To determine the vertices A, *a* of the mountains, we have, on the west, the base $BC = 3$ chains, and the lines AB, AC equal 2.90, and 2.65 chains respectively; and, on the east, the base *bc*, and the lines *ab*, *ac* equal 3.05, and 3.45 chains respectively.

To determine the form of the mountains, the lines AB, AC, *ab*, *ac*, &c. are laid off from the summits A, *a*, respectively, down the sides. The secant planes are taken at a distance of 10 links, or feet, from each other. Passing from the top downward, the distances between the points at which the secant planes come out upon the lines are noted in the following tables.

WEST MOUNTAIN.

AB	AC	AD	AE	AF	AG	AH
.20	.15	.20	.15	.18	.20	.30
.30	.35	.30	.25	.20	.30	.40
.60	.75	.80	.60	.55	.50	.60
.50	.40	.65	.50	.40	.40	.80
.40	.20	.80	.70	.50	.35	.90
.90	.80	.95	.60	.60	.80	1.25

EAST MOUNTAIN.

ab	*ac*	*ad*	*ae*	*af*	*ag*	*ah*
.30	.25	.19	.25	.20	.25	.40
.45	.40	.40	.50	.20	.30	.25
.60	.75	.80	.80	.40	.50	.80
.70	.50	.50	.40	.50	.20	.30
.45	.85	.90	.60	.65	.60	.95
.25	.30	.70	.75	.30	.50	1.00
.30	.40	.55	.80	.40	.30	.50

The angles between the lines AC, AD, &c. are laid off from AC through the north and west round to AB, and are as follows, viz. $CAD = 41^\circ$, $DAE = 44^\circ$, $EAF = 60^\circ$, $FAG = 45^\circ$, $GAH = 60^\circ$.

The angles between the lines *ac*, *ad*, &c. are laid off from *ac* through the north and east round to *ab*, and are as follows, $cad = 40^\circ$, $dae = 43^\circ$, $eaf = 42^\circ$, $fag = 50^\circ$, $gah = 61^\circ$.

* * *

SECTION VII.

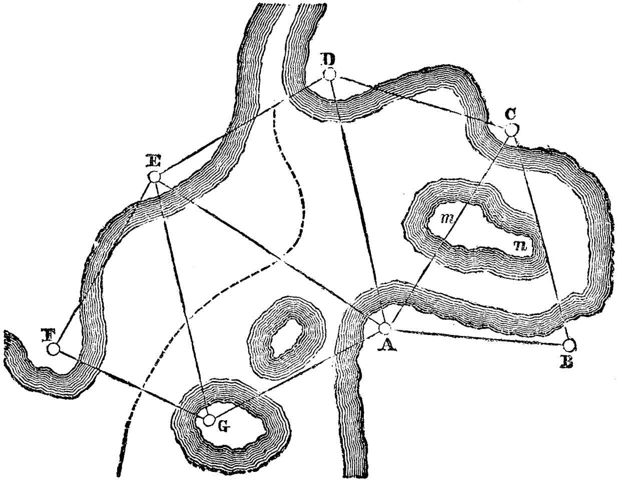
MORE EXTENDED SURVEYS.

173. The surveys, thus far, have been fields or portions of the earth's surface of small extent. We proceed to explain the method to be pursued in more extended surveys, such as harbors, towns, counties, &c.

SURVEYING HARBORS.

174. In surveying harbors the object is, 1°, To trace the shore along the line of high or low water mark, taking care to notice all the prominent points, such as inlets, places where streams discharge themselves into it, &c. 2°. To determine the depth of the water, to mark the channel, the position of rocks, shoals, &c., or whatever may affect navigation.

Fig 68.



Let the preceding figure (fig. 68) represent the harbor to be surveyed.

We select the more prominent points A, B, C, D, &c. as vertices of a series of triangles ABC, ACD, &c, covering the entire extent of the harbor. We next measure all the angles in this series; and, selecting the most convenient, we measure also a side in one of the triangles, from which, by means of the angles the remaining sides of the triangle, as well as those of the rest of the series, may be determined.

To mark the position of other points, we take the bearings of these points from the extremities of a side in some one of the principal triangles. Or, where greater accuracy is required, we measure the angles which lines drawn from these points to the extremities of a side in one of the triangles make with this side. Thus, if we wish to mark with accuracy the extremities *m* and *n* of the island *mn*, we measure the angles *m*AB, *m*BA, *n*AB, *n*BA, which lines drawn from *m* and *n* to A and B respectively make with the side AB.

To trace the shore we commence with the chain and compass at F, for example, and run lines as near the water's edge as possible, following all the bends and indentations, and making offsets when necessary for a more accurate delineation of the line of the shore. Between the points thus determined, a sketch of the sinuosities of the shore line may be made in the field book in the manner explained for the chain and compass work in leveling.

The chain and compass work should always connect with the vertices of the principal triangles, so that any small errors in the work between any two of these points may not be propagated beyond them.

To determine the depth of water various methods are employed.

1. In lines at suitable distances from each other, buoys

may be anchored across the harbor, and their positions determined by their bearings from the vertices of the triangles or other fixed points. The depth of the water may then be taken at the buoys.

2. Instead of the buoys a boat may be anchored successively at different points along the lines of the soundings, and its position in each case being determined, the depth of the water may be taken as at the buoys.

3. Or on the lines on which soundings are to be taken, a boat may be rowed with a slow and uniform motion, and the depth of the water taken at equal intervals of time. This is, however, a rough and approximate method, and the former methods are preferable.

The depth of water at the soundings being marked upon the plan, a dotted line drawn through the points of deepest water will indicate the channel.

From what has been done it will readily be seen how we are to mark the position of shoals, rocks, &c., whether covered with water or not, as well as other points it may be important to notice. The topography of the harbor may also be noted in the manner already explained.

175. From what has been done the course to be pursued in an extended survey is sufficiently obvious.

1°. The surveyor will carefully examine the whole territory to be surveyed, and set up flag-staffs at all the prominent points to serve as signals for the determination of the position of these points.

2°. He will next select the most important of these to serve as vertices to a system of triangles connected with each other, and forming a net work covering the whole ground.

3°. All three of the angles in each of these triangles must be measured with the theodolite, the third angle being measured to test the accuracy of the other two.

4°. A side of one of the triangles must be measured as a base line; or, where this is not convenient, a base line must be measured in some convenient place, and so connected with the principal triangles that they may be computed from it.

5°. The principal points being thus determined, as many other minor points as we please may be determined by triangles laid off from the sides of the principal triangles.

6°. The details of the survey will now be filled in with the chain and compass, and the topography sketched as heretofore explained.

The method of determining a series of points by means of triangles is called a *triangulation.*

The work of determining the principal triangles is called the *primary* triangulation, that of the triangles of less importance laid off from these the *secondary* triangulation.

In addition to the primary base line, which we have supposed to be measured near one extremity of the triangulation, it is usual to measure another near the other extremity. This being calculated from the first through the series of triangles, the agreement of the calculated with the measured base will furnish a test of the accuracy of the triangulation.

PLANE TABLE.

176. A very convenient instrument for mapping the details of a survey is the plane table, the construction and use of which we will briefly explain.

The principal parts of the plane table are a board about 16 inches square and a tripod for its support.

The table is bordered by brass plates about an inch in width, and its centre is marked by a pin. By means of lines drawn from the centre, the outer edges of the plates are graduated so as to represent degrees, half degrees, &c. The degrees are numbered from left to right from 0 to 180°. The

0 is placed at one extremity of a diameter drawn through the centre parallel to one of the sides, and 180° at the other.

Two brass plates are fitted into grooves on opposite sides of the board, which, by means of screws from beneath may be raised so as to permit a sheet of paper to be laid upon the board with its edges beneath them. The plates then being screwed down, the paper is drawn tight across the board and thus secured to it.

The table is attached to the tripod by a ball and socket joint, with brass plates and screws for the purpose of leveling it. This is done by aid of a detached spirit level.

Finally the instrument is fitted with a movable ruler about the length of the diagonal of the board, with sight vanes like those of the compass screwed on to each end. The fiducial edge of the ruler is in the same plane with the vertical threads and slits of the sight vanes.

The plane table may be used for the measurement of horizontal angles. Its principal use, however, is for determining the shorter lines of a survey both in extent and position.

Thus, let it be supposed that we wish to trace the southern shore of the island *mn* (fig. 68) from the base line AB.

We place the instrument at A, with a sheet of paper upon it, so that the larger part of the paper shall be toward the shore, and bring the instrument to a level. By means of a plumb line we then stick a pin in the board directly over A. Placing the edge of the ruler against the pin we sight to the end B of the base line, and draw a pencil line along the edge of the ruler toward B. Suppose a series of stakes driven along the shore and numbered 1, 2, 3, 4, &c. We place the edge of the ruler against the pin, and sight in succession to the stakes, and draw lines in pencil indefinitely toward them, numbering them 1, 2, 3, 4, &c. We next lay off, by means of a convenient scale of equal parts, on the line drawn toward B, a line equal in length to the base AB, and place a pin at the end. We then remove the instrument to the station B, and bring it to a level, with the pin last stuck in directly over B; and, by means of the ruler, we arrange the line between the two pins on the board in the direction of the base BA. We then place the edge of the ruler against the pin over B, and sighting back to the stakes 1, 2, 3, &c. we draw lines toward them intersecting those before drawn. The points of intersection will be the place of the stakes as represented on the board; and, connecting these points by lines, we shall have a trace of the shore upon the scale used in laying down the distance between the two pins, or base line on the board corresponding to the base line AB.

177. The following example will serve as an illustration of an extended survey. The territory surveyed is bounded on the south by the Atlantic Ocean. The work was commenced in the most westerly part, and the series of triangles, by which the prominent points are connected, extends northerly and easterly to the last triangle, the vertex of which forms the most southerly point of the survey.

PRIMARY TRIANGULATION.

Triangles.	Angles.	Sides.	
I. ACB	ACB = 59°	AB = 77.54 ch.	Base.
	ABC = 56°	AC = 75.00	
	BAC = 65°	BC = 81.99	
II. BCD	BCD = 68° 30′	BD = 88.99	
	CBD = 52° 30′	CD = 75.88	
III. BDE	BDE = 62°	BE = 90.73	
	DBE = 58°	DE = 87.15	
IV. BEF	BEF = 52° 30′	BF = 74.89	
	EBF = 53° 30′	EF = 75.88	
V. FEG	FEG = 51°	EG = 82.48	
	GFE = 69° 30′	FG = 68.43	
VI. FGH	FGH = 56°	FH = 67.27	
	HFG = 66° 30′	GH = 74.48	

The side AB in the first triangle was measured. It forms the base of the survey. From this, for the sake of greater accuracy in plotting, all the other sides are computed. The bearing of the base is North 75° East. The stations, or vertices of the triangles, are lettered A, B, C, D, E, F, G, and H.

The stations B and D are on the westerly bank of a river which runs in a southerly direction through the territory. The station B is also near the mouth of a branch of the river which comes in from the northwest, passing by and near the station C.

CHAIN AND COMPASS WORK.

1. Commencing at B and running to C; thence crossing to P and running to D. P is a stake on the opposite side of the river from C, distant 7 chains from C in the direction CD.

Sta.	No.	Bearing.	Dist.	L. Offsets.	R.
to D.			34.50		2.50
			17.00	.00	
	4.	N. 17½° W.	6.00	1.50	
			34.00		.00
			24.00		4.00
			12.00		5.00
	3.	N. 1½° E.	.00		2.00
			41.00		2.00
			23.00		1.00
	2.	S. 81½° E.	7.00		3.00
			43.00		3.00
			32.00		1.00
			14.00		5.00
P.	1.	S. 48½° E.	.00		.50
to C.			33.50		4.00
	2.	N. 42° W.	20.00	1.50	
			49.00		4.00
			13.00		.00
			9.00		5.00
B.	1.	N. 54½° W.	.00		3.00

2. Commencing at H and running through F and M to N. M is a stake on the opposite bank of the river from B, distant 25 chains from B in the direction BF. N is also a stake on the opposite bank of the river from D, distant 12 chains from D in the direction CD produced.

Sta.	No.	Bearing.	Dist.	L. Offsets.	R.
to N.			46.00	.00	
			35.00	2.00	
			30.00		.00
	3.	N. 24° W.	18.00		1.00
			33.50	2.00	
			26.00		1.50
	2.	N. 4° W.	10.00	.00	
			32.00	1.50	
			23.00		2.00
M.	1.	N. 27° E.	13.00	.00	
to M.			23.50	1.00	
			16.00	5.00	
			10.00	1.50	
			5.00	4.00	
	2.	N. 7° W.	.00	2.50	
			42.00	2.00	
			32.00		4.00
			20.00	.00	
F.	1.	S. 87° W.	14.00	3.00	
to F.			35.50	2.00	
			27.00	6.00	
	2.	N. 54° W.	18.00	2.00	
			35.00		.00
			28.00		3.00
			23.00	.00	
			10.00	8.00	
H.	1.	N. 17° W.	.00	2.00	

3. From N the following courses were run on the shore of the river, viz. 1, N. 29° E. 23 chains; 2, N. 68° E. 27 chains; 3, N. 37° E. 30 chains.

4. From B to A. Courses run and offsets to the shore.

1. S. 45° W. 56 chs. Offsets—at 0,3 chs. L; at 15 chs. 5 chs. L; at 27 chs. 0; at 42.chs. 4 chs. R; at 50 chs. 0; at 56 chs. 2 chs. L. 2. N. 62° W. 41 chs. Offsets—at 2 chs. 3 chs. L; at 20 chs. 2 chs. R; at 41 chs. 8 chs. L.

5. A village is situated on the point of land between the main river and its branch. Its northern boundary is determined as follows. Commence at P and run N. 17° W. 22 chains to a monument at the water's edge. This marks the north-west corner of the village. The line runs thence east to the main river.

The village is laid out in streets which run north and south, east and west.

SECONDARY TRIANGULATION.

1. The principal peaks of a mountain range north and east of the main triangulation are determined by the following angles. The peaks are marked P_1, P_2, P_3, &c. in order. The sides DE, and EG in the main triangulation form the bases of the secondary triangles which have their vertices at P_1, P_2, P_3, &c.

Station D.	Station E.	Station G.
$P_1\,DE = 90^\circ$	$P_1\,ED = 37^\circ$	$P_5\,GE = 22^\circ$
$P_2\,DE = 65^\circ$	$P_2\,ED = 32^\circ$	$P_6\,GE = 54^\circ$
$P_3\,DE = 41^\circ$	$P_3\,ED = 61^\circ$	
$P_4\,DE = 30^\circ$	$P_4\,ED = 75^\circ$	
	$P_5\,EG = 69^\circ$	
	$P_6\,EG = 47^\circ$	

2. An island off the mouth of the river is determined as follows. Two stations *m* and *n* are taken at the extremities of the island forming with the stations A and B the secondary triangles AB*m*, AB*n*.

Station A, Angles, $BAm = 62^\circ$; $BAn = 22\frac{1}{2}^\circ$

" B " $ABm = 47^\circ$; $ABn = 67^\circ$

The chain and compass work for the island is as follows.

I. From *m* to *n*. 1. N. 77° E. 26 chs. Offsets—at 10 chs. 5 chs. R; at 13 chs. 1 ch. R; at 20 chs. 3 chs. R. 2. N. 9° E. 29 chs. Offsets—at 13 chs. 2 chs. L; at 20 chs. 1.50 chs. R.

II. From *n* to *m*. 1. S. 47° W. 42 chs. Offsets—at 5 chs. 3 chs. R; at 10 chs. 2 chs. R; at 20 chs. 4 chs. R; at 32 chs. 1 ch. L. 2. To *m* 6 chs.

3. The position of a light house L, south of the main triangles, is determined by the following angles in the secondary triangle BFL; viz. LBF = 63°, LFB = 70°.

POINTS DETERMINED BY COMPASS.

1. From the stations F and H a reef of rocks is determined by the following bearings.

	Station F.		Station H.
R_1	S. 31° W.	R_1	N. 60° W.
R_2	S. 23° W.	R_2	N. 72° W.
R_3	S. 16° W.	R_3	N. 81° W.
R_4	S. 5° W.	R_4	N. 87° W.

2. A rock in the main river is determined by the following bearings from the stations B and M, viz. from B, N. 74° E; from M, N. 1° W.

3. Through a point K 5 chains west of the station F passes the easterly line of a road, which runs north and south through the territory surveyed. Commencing at K the territory is laid out, by east and west lines, into farms 40 rods in width. The farms extend on the west of the road to the river, and are bounded on the east by a north and south line distant 50 chains east from the station F.

TOPOGRAPHY.

The learner will sketch the shores of the river, &c. by the proper character. The mountain range, the peaks of which have been determined, will also be sketched by the aid of irregular curves, as their sides are regarded as more or less steep, as if determined in the manner explained art. 167. The points of the principal triangles or primary stations are all situated upon elevated ground which may be indicated in the same manner.

On the east side of the road which passes through K, are situated the farm houses belonging to the farms into which the territory is divided. On this side also are situated the pastures and wood lots. Between the road and river are the gardens, orchards, cultivated fields, grass ground, &c. The island is covered with pine trees.

Let the learner draw at pleasure the village on the west side of the river, laying off the adjacent lands into farms, country seats, &c.

The careful filling up of the map will be found a useful exercise. What has been done is sufficient to show how the details may all be connected with the main triangulation. This is the most important part of the survey and should always be performed with the greatest care.

TRIGONOMETRICAL SURVEYING.

178. By the process of triangulation, above explained, places however distant from each other may be connected, and the geographical position of any one of them being known, that of the others may be found from it. The method of determining the position of places, at a considerable distance from each other, by aid of a series of triangles is called *Trigonometrical Surveying.*

By this method we are enabled to determine the length of a degree upon a meridian or true north and south line, and thus ascertain the whole circumference of the earth considered as a sphere.

To give an elementary idea of the manner in which this work is performed, let the distant points A, and E (fig. 69) be connected by a series of triangles ABC, BCD, &c. Let A*b* considered as the *base line*, be measured at one extremity of the series, and D*f* as a *base of verification* at the other extremity; and let all the angles in the series be measured. We thus obtain the means of calculating the sides AC, C*n*, *n*E of the series.

Next, by the method explained arts. 147, &c. let the azimuth, or bearing from a true north and south line, of one of the sides AC be determined. We shall then, with the angles previously measured, be able to find the azimuth of the remaining sides C*n*, *n*E. We thus have the means for calculating the true differences of latitude M*m*, *mn*, &c. of the points A and C, C and *n*, &c. from each other; from which we obtain the whole difference of latitude MN between the extreme points A and E. This difference will be expressed in feet, yards, or miles, or in whatever denomination of linear measure we may adopt in the measurement of the base line.

Finally by astronomical observations the exact latitude of the points A and E must be found, by which we shall know the precise difference of latitude in degrees between A and E. Comparing this with the difference of latitude between these two points found by the survey, the length of a degree upon an arc of the meridian will be known, and thus the whole circumference of the earth considered as a sphere may be found.

Thus in the great French Survey extending from the British channel to the Mediterranean, the difference of latitude between the extreme north and south points of the survey is 12°.370303; the whole length of the arc as determined by the triangulation is 751567 English fathoms or 4509402 feet. This gives, 69.0405 miles nearly for the length of a degree; which, being multiplied by 360 gives 24854.58 miles for the circumference of the earth considered as a sphere.

By measurements of this kind the form of the earth may be determined. From measurements already made in Peru, India, France, England and Sweden, it is found that the degrees increase in length as we proceed from the equator to the pole. It is thus ascertained that the earth is not a sphere but is flattened toward the poles.

Ex. 1. In Peru the latitudes of the extreme points of the survey are $0^\circ\ 02'\ 31''.22$ S. and $3^\circ\ 04'\ 31''.9$ N. respectively, and the measured arc comprised between them 1131060 feet. What is the length of a degree derived from this measurement? Ans. 68.7132 miles.

Ex. 2. In the survey in Sweden the difference of latitude between the extreme parallels was found to be $1^\circ.622022$, and the length of the measured arc 593220 feet. What is the length of the degree? Ans. 69.2668 miles.

DIVISION OF LANDS.

179. In the irregular field A B C D E F (fig. 70) let it be required to draw a line from the corner A, so as to divide the field into two equal parts.

From inspection of the plot it appears that the dividing line AX will fall upon the side DE. Run the line AD from A to D, and calculate the part ABCD which suppose less than one half of the field. Subtracting this area from one half the area of the field we shall have the area of the triangle ADX, which must be added to ABCD to effect the division required. To find this triangle we have the side AD, the angle ADX, and the area A. Putting x for DX the base of the required triangle we have

$$\tfrac{1}{2}x \times \mathrm{AD} \times \sin \mathrm{ADX} = \mathrm{A}$$

whence $$x = \frac{2\mathrm{A}}{\mathrm{AD} \sin \mathrm{ADX}}$$

BOOK III. NAVIGATION.

SECTION I.

DEFINITIONS. DESCRIPTION OF INSTRUMENTS.

180. Navigation is the art of conducting a ship from one port or place on the earth's surface to another.

The place of a ship at any time is determined by her latitude and longitude.

Latitude is distance from the equator measured on a meridian, and is either north or south.

Longitude is distance from an assumed meridian measured on the equator, and is either east or west.

Departure is the distance east or west between two places measured on a parallel of latitude.

There are two ways in which the place of a ship at sea may be determined.

1°. By keeping account of the course and distance sailed, and deducing therefrom her latitude and longitude.

2°. By calculating her latitude and longitude from observations upon the heavenly bodies.

It is the former method only which we shall consider.

THE MARINER'S COMPASS.

181. The instrument by which the course of a ship is determined is the *Mariner's Compass.*

It is the same in principle with the Surveyor's Compass. Its principal parts are 1°, A circular card, the circumference of which is divided into 32 equal parts called *points*, and these again into four equal parts called *quarter points.* 2°. A magnetized needle placed beneath the card, and so attached to it that the centre on which the needle turns is at the centre of the card, and its north and south points directly under the points marked N. and S. on the face of the card. 3°. A compass box with adjustments such that the card, when suspended on its pivot, will always maintain a horizontal position whatever the position of the ship.

The compass is so placed on ship-board that its centre is in a vertical plane passing from stem to stern through the centre of the ship. The needle having a free motion on its centre, the point of the compass directed to the ship's head will indicate her course.

182. The points of the compass, commencing at the north and passing round from right to left, are marked on the compass card, as follows.

1st Quad.	2d Quad.	3d Quad.	4th Quad.
N.	E.	S.	W.
N. by E.	E. by S.	S. by W.	W. by N.
N. N. E.	E. S. E.	S. S. W.	W. N. W.
N. E. by N.	S. E. by E.	S. W. by S.	N. W. by W.
N. E.	S. E.	S. W.	N. W.
N. E. by E.	S. E. by S.	S. W. by W.	N. W. by N.
E. N. E.	S. S. E.	W. S. W.	N. N. W.
E. by N.	S. by E.	W. by S.	N. by W.

Beginning at the north and passing round in order, the points of the compass are read as follows; north, north by east, north-north-east, north-east by north, north-east, north-east by east, east-north-east, east by north, east, &c.

To begin at any point and name the points of the compass in order is called by seamen *boxing the compass.*

183. A point being one-eighth of a quadrant is equal to 90° ÷ 8, or 11° 15′. A quarter point is, therefore, 2° 48′ 45″. In running a course mariners do not take into account an angle less than a quarter point.

In the following table we have the number of degrees and minutes corresponding to each point of the compass.

North.		Pts.	Deg.	South.	
N. by E.	N. by W.	1	11° 15′	S. by E.	S. by W.
N. N. E.	N. N. W.	2	22° 30′	S. S. E.	S. S. W.
N. E. by N.	N.W. by N.	3	33° 45′	S. E. by S.	S. W. by S.
N. E.	N. W.	4	45°	S. E.	S. W.
N. E. by E.	N.W. by W	5	56° 15′	S. E. by E.	S.W. by W.
E. N. E.	W. N. W.	6	67° 30′	E. S. E.	W. S. W.
E. by N.	W. by N.	7	78° 45′	E. by S.	W. by S.
East	West	8	90°	East	West

THE LOG.

184. The rate of a ship's sailing is determined by the log. This consists of a board in the form of a sector of a circle, and loaded at the circumference so as to maintain a vertical position in the water. A line called the *log-line* is attached to it in such a manner that the broad surface, or face of the log, is kept toward the ship. Thus, by the resistance of the water, the log is kept in its place, while the line is run off from a reel, as the ship moves forward.

The length of line run off in a *half minute* gives the rate of sailing. The log line is usually divided into equal parts called *knots.* Each knot is taken equal to the 120th part of a geographical mile or degree. And since a half minute is the 120th part of an hour, the number of knots and parts of a knot run off in a half minute will express the number of miles and parts of a mile which the ship makes an hour, at the same rate of sailing.

The process of determining the rate of sailing by the log is called by seamen heaving the log.

RHUMB LINE.

185. The line of rhumbs on Gunter's scale is constructed in a manner similar to the line of chords. The quadrant is divided into eight equal parts corresponding to the points of the compass, and these again into four equal parts for the quarter points. The chords of these several divisions are then transferred to the same line, and are duly marked and numbered, as in the case of the line of chords.

The use of the line of rhumbs is to lay down the course of a ship in points and quarter points of the compass. The radius of the line of rhumbs will be the chord of 60°.

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SECTION II.

PLANE SAILING.

186. For short distances the earth's surface may, without sensible error, be considered as a plane, and the path of a ship, her course, difference of latitude and departure may be regarded as parts of a right-angled plane triangle.

Ex. A ship in latitude 37° 10′ N. sailed N. N. E. 80 miles. What is the departure made, and the latitude of the ship?

1°. *By construction.* We draw an indefinite line NS, (fig. 71) to represent the meridian from which the ship departs. Assuming the point A as the place of departure, we lay off by aid of the line of rhumbs the angle NAB equal to the course, which in this case is 2 points. Through this we draw $AB = 80$ the distance sailed, and from B let fall BC upon the line NS; AC will be the difference of latitude and BC the departure, which may now be measured upon the same scale of equal parts from which the distance AB was laid down.

2°. *By computation.* The same figure being supposed we have

As rad.	10.000000
: Distance 80	1.903090
: : Sin Course 22° 30′	9.582840
: Departure 30.61	11.485930

Thus the departure is 30.61 miles. In like manner we find the difference of latitude = 73.91 miles. Dividing this by 60, since sixty geographical miles make a degree, we have 1° 13′ 55″ for the difference of latitude in degrees. This being added to 37° 10′, the latitude of the point of departure, gives 38° 23′ 55″ for the latitude of the ship.

187. The right angle being always known, of the four things, course, distance, difference of latitude, and departure, the others may be found when any two of them are given. The following are examples of the cases which may occur.

I. Given the course and distance.

Ex. A ship sails from latitude 47° 30′ N. upon a course S. W. by S. 98 miles. What latitude is she in, and what departure has she made?

Ans. Lat. in 46° 8′ N. Dep. 54.45 miles.

II. Given the course and difference of latitude.

Ex. A ship in latitude 23° 7′ S. sails E. N. E. until she comes to the latitude of 22° 10′ S. What is the distance sailed and departure? Ans. Dist. 149 miles. Dep. 137.6 miles.

III. Given the course and departure.

Ex. A ship in latitude 16° 35′ S. sailed on a course N. E. ½ N. until she found her departure to be 342 miles. What was the distance sailed and the latitude at which she arrived? Ans. Dist. 540 miles. Lat. 9° 38′ S.

IV. Given the distance and difference of latitude.

Ex. A ship from latitude 1° 4′ S. sails between the N. and W., that is, north-westerly, 162 miles, and then finds her latitude by observation to be 52′ N. Required her course and departure. Ans. Course N. 44° 15′ W. Dep. 113 miles.

V. Given the distance and departure.

Ex. A ship sails on a southerly course 118 miles, and makes 83 miles westing. Required her course and difference of latitude. Ans. Course S. 44° 42′ W. Diff. Lat. 83.8 miles.

VI. Given the difference of latitude and departure.

Ex. A ship in latitude 30° 5′ north sails southwesterly until she has made a difference of latitude of 13.3 miles, and her departure is 19.9 miles. What is her course and distance sailed? Ans. Course S. W. by W. Dist. 24 miles.

188. The process above is true not only for short distances in which the surface of the earth may be considered as a plane, but for all distances when the course is unchanged.

To demonstrate this let AD (fig. 72) represent the path described by the ship or the distance sailed, P the pole of the earth, PN and PM the extreme meridians passing through A and D, NM a portion of the equator, and DAS the angle of the course.

Since the ship makes the same angle with each meridian she passes while sailing constantly on the same course, her path, it is evident, will be a curve line. This curve moreover is not a portion of a great circle, but is a peculiar kind of curve, called a *rhumb-line* or *loxodromic curve.*

Suppose next AD divided into any number of equal parts so small that any one of them, CE for example, may be regarded as a straight line. Through C and E draw the meridians PK, PL, and draw CF parallel to NM. Through A and D draw also the parallels AO and DS. The triangle CEF will be right angled at F and may be regarded as a plane triangle. In like manner let triangles be conceived to be constructed on all the parts of AD equal each to CE. The sum of the parts CE, or the sum of the hypothenuses of the small triangles will, it is evident, be equal to AD or the distance sailed, and the sum of the sides EF will be equal to AS the difference of latitude ; and we shall have the proportion,

sum of the sides CE : sum of the sides EF : : CE : EF

or AD : AS : : CE : EF (1)

Let next a right angled triangle (fig. 73) be constructed in which the hypothenuse AB is taken equal to AD the distance sailed by the ship, and the angle BAC equal to the course DAS. This triangle will be similar to CEF, and we have

AB : AC : : CE : EF (2)

whence comparing (1) and (2)

AD : AS : : AB : AC (3)

in which if AB is equal to AD, AC will be equal to AS.

Notwithstanding, therefore, the path described by the ship is a curve line, if we construct a right angled plane triangle, in which the hypothenuse is equal to the distance sailed and one of the acute angles equal to the course, *the side adjacent to this angle will be equal to the difference of latitude whatever the distance sailed.*

By a process altogether similar it may be shown that the side BC, or that opposite the course, will be equal to the sum of the departures CF or the whole departure.

We thus see that all the cases of plane sailing, in which the distance, course, difference of latitude and departure are the only things considered, may be solved by aid of a right angled plane triangle.

189. Thus far we have considered the place of a ship determined by the difference of latitude and departure. To know, however, her actual position we must be able to convert

the departure into difference of longitude, in order that her longitude may also be found.

Suppose that a ship sails from D to S (fig. 72) on the parallel of latitude DS. In this case the departure DS will be equal to the distance sailed; and to determine the actual change of place of the ship, east or west, we must find the difference of longitude, or length of the arc NM on the equator, corresponding to the departure DS.

In order to this it will be recollected that similar arcs are as their radii; whence DS is to NM, as the radius with which DS is described to the radius with which NM is described. Regarding the latter as R, or the sine of 90°, the radius of the tables, the former will be the sine of DP, or the cosine of ND; whence

$$\cos ND : R :: DS : NM,$$

or, *The cosine of the latitude is to radius as the departure is to the difference of longitude.*

190. When a ship sails exactly east or west on the same parallel, the only change made in her position is in the longitude which may be correctly determined by means of the above proportion. This case is usually called *parallel sailing.*

Ex. 1. A ship sailed from Boston in longitude 71° 4′ 9″ W. 1000 miles exactly east; what is the longitude at which she has arrived? Ans. 48° 32′ W.

Ex. 2. What is the departure in the latitude of 45° corresponding to one degree of longitude? Ans. 42.43 miles.

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SECTION III.

MIDDLE LATITUDE SAILING.

191. A ship ordinarily changes both her latitude and longitude at the same time, and the question returns in this case

to find the difference of longitude. Thus suppose AD (fig. 72) the distance sailed on an oblique course DAS. We have seen that AB (fig. 73) being taken equal to the distance AD, and the angle BAC = DAS, CB will be equal to the sum of the departures CF, or to the whole departure. But on what parallel shall this departure be reckoned ? It is evident that it cannot be reckoned on AO, the parallel passing through A the point from which the ship departs; for, in consequence of the convergency of the meridians, the sum of the sides CF is manifestly less than AO; nor can it be reckoned on the parallel DS, the parallel passing through D the point at which the ship has arrived; for the sum of the sides CF, for the same reason as before, is greater than DS. On the whole it will be sufficiently near in general to reckon the sum of the sides CF, or the departure, upon the middle parallel VT, or that which passes half way between AO and DS. It is this in which the principle of middle latitude sailing consists.

Upon CB in the triangle CAB construct the triangle CBD in which the angle CBD is taken equal to the middle latitude; then BD will be equal to the difference of longitude, since we have sin CDB or cos CBD : R : : CB : BD

192. The two triangles CAB, CBD constitute together a single triangle ABD. From the three we obtain the following proportions.

I. From the triangle CBD

sin CDB, or cos CBD : CB : : R : BD;

or in words, *The cosine of the middle latitude is to the departure, as radius to the difference of longitude.*

II. From the triangle ABD

sin ADB : AB : : sin BAD : BD;

or, *The cosine of the middle latitude is to the distance, as the sine of the course to the difference of longitude.*

III. From the triangle BAC

$$R : \tan BAC :: AC : CB;$$

comparing this with the first proportion, and observing that the extremes in this last are the means in the first, we obtain

$$AC : BD :: \cos CBD : \tan BAC,$$

or, *The difference of latitude is to the difference of longitude, as the cosine of the middle latitude to the tangent of the course.*

193. The three proportions above comprise all that is necessary for the solution of the different cases in middle latitude sailing. We subjoin a few examples, observing that in these, as well as in the remaining examples of this work, longitude is reckoned from the meridian of Greenwich.

I. Given the latitude and longitude of the place of departure, and the course and distance.

Ex. A ship in latitude 47° 23′ N., and longitude 10° 17′ W. sails 126 miles on a course S. W. by W. What is the latitude and longitude of the place at which she arrives?

We first find, as in plane sailing, the difference of latitude = 70 miles, or 1° 10′. This being subtracted from 47° 23′, the latitude left, gives 46° 13′ for the latitude in.

Adding next half of 1° 10′ or 35′ to the less latitude 46° 13′, we obtain the middle latitude 46° 48′. Then by the second proportion we obtain the difference of longitude equal to 153 miles or 2° 33′. And as the direction in which the ship sailed is from the first meridian, adding this to 10° 17′, the longitude of the place of departure, we obtain 12° 50′ W. for the longitude of the place at which she arrives.

II. Given both the latitudes and longitudes.

Ex. A ship in latitude 29° 40′ S. and longitude 53° W. sails on a northwest course until by observation her latitude is found to be 26° 13′ S. and longitude 55° 38′ 36″ W. What is her course and the distance sailed?

Ans. Course N. 34° 6′ W. Dist. 250 miles.

III. Given the latitude and longitude of the place left, the latitude in and the departure.

Ex. A ship in latitude 38° 10′ N. and longitude 56° W. sails on a southeasterly course until she is found by observation to be in latitude 37° 11′ N. having made 54 miles departure. What is the longitude in and the course on which she has sailed ? Ans. Long. 54° 51′ 48″. Course S. 42° 28′ E.

IV. Given the latitude and longitude left, the latitude in and the course.

Ex. A ship in latitude 42° 30′ N. and longitude 41° 45′ W. sails S. E. by S. until she finds her latitude to be 38° 21′ N. Required the distance sailed and longitude in.

Ans. Dist. 299.5 miles. Long. 38° 6′ 30″ W.

V. Given the longitude left, both latitudes and the distance.

Ex. A ship in latitude 23° 11′ N. and longitude 53° 38′ W. sails on a northwesterly course 230 miles, when she finds her latitude by observation 25° 43′ N. Required her course and longitude in.

Ans. Course N. 48° 38′ W. Long. 56° 47′ 36″.

VI. Given the latitude and longitude left, the course and departure.

Ex. From a port in latitude 53° 7′ N. and longitude 63° 15′ W. a ship sails E. S. E. until her departure is 112 miles. What is the latitude and longitude in.

Ans. Lat. 52° 21′ N. Long. 60° 10′ 6″ W.

VII. Given the latitude and longitude left, the distance and departure.

Ex. A ship from Sandy Hook in latitude 40° 28′ N. and longitude 74° 1′ W. sails southeasterly 123 miles having made 103 miles departure. Required the place of the ship.

Ans. Lat. 39° 12′ N. Long. 71° 46′ 54″ W.

The method of middle latitude sailing gives the difference of longitude with sufficient accuracy for moderate distances near the equator, where the convergency of the meridians is small. It becomes increasingly more inaccurate as we approach the poles. A correction, however, may be applied which will render the process correct for all latitudes. Tables for this purpose have been prepared by Workman.

SECTION IV.

TRAVERSE SAILING.

194. Thus far we have supposed the ship to sail on a single course. But ordinarily, in going from one place to another, a ship sails on several different courses. And the question is to find the single course and distance equivalent to the several courses and distances which the ship has run.

The irregular track made by the ship is called a *Traverse;* and the finding the single course and distance equivalent to it, is called compounding or *working the Traverse.*

To find the whole difference of latitude made by the ship we must find, it is obvious, the several differences of latitude made upon the courses run separately; then subtracting the sum of the differences corresponding to the southerly courses from the sum of the differences corresponding to the northerly courses, or the converse as the case may require, we shall have the difference of latitude on the whole made by the ship; or, as it is sometimes expressed, the difference of latitude made good. In like manner we find the departure on the whole made by the ship. With the difference of latitude and departure thus found we construct a right angled triangle, the hypothenuse of which will be the equivalent distance, and the acute angle adjacent to the difference of latitude the equivalent course sought.

Ex. 1. A ship from a certain port in latitude 42° N. sails on the following successive tracks; 1, S. S. E. 80 miles; 2, S. E. 55 miles; 3, S. W. by W. 115 miles; 4, E. S. E. 62 miles. Find the course and distance for the whole traverse.

1°. *By construction.* With the chord of 60° from the line of chords, we draw a circle (fig. 74) to represent the horizon, and which we call the horizon circle. Through the centre A of this circle we draw indefinitely a line NS, to represent the meridian passing through A the point of departure. We next, by aid of the rhumb line, lay off on the horizon circle the several courses, numbering them 1, 2, 3, &c. in order. In the present case the courses being all southerly are laid off from the south point S, those which are easterly to the right, and those which are westerly to the left. This being done, from A in the direction A1 we draw AB equal 80 miles the distance sailed on the first course; next from B in a direction parallel to A2 we draw BC equal 55 miles the distance sailed on the second course, and so on to DE the distance sailed on the last course. Then E will be the place of the ship at the end of the traverse. Letting fall next from E the perpendicular EF, AF will be the difference of latitude and FE the departure made good. Joining AE, AE will be the distance and FAE the course equivalent to the traverse.

2°. *By computation.* It is convenient to arrange the data and calculations in a tabular form analogous to what we have done in surveying. Thus, for the present example, we have the following table which sufficiently explains itself.

No.	Course.	Dist.	N.	S.	E.	W.
1	S. 22° 30′ E.	80 mls.		73.9	30.6	
2	S. 45° E.	55		38.9	38.9	
3	S. 56° 15′ W.	115		63.9		95.6
4	S. 67° 30′ E.	62		23.7	57.3	
Sum of columns				200.4	126.8	95.6

Subtracting the westing from the easting we obtain 31.2 miles for the departure made good. From this with 200.4 miles the difference of latitude made good, we find, by plane sailing, the course S. 8° 51′ E., and the distance 202.8 miles.

195. The course and distance being thus found, the difference of longitude may next be found by middle latitude sailing.

Ex. 2. A ship sails on the following successive courses; 1, S. S. E. 31 miles; 2, S. E. by E. 27 miles; 3, S. W. 42 miles; 4, W. by N. 34 miles; 5, S. by E. 50 miles. What is the single course and distance equivalent?

Ans. Course S. 9° 45′ 33″ W. Dist. 117.4 miles.

Ex. 3. A ship is bound to a port distant 100 miles from the place of departure and bearing W. by S. She has sailed on the following courses; 1, S. 10 miles; 2, W. S. W. 25 miles; 3, S. W. 30 miles; 4, W. 20 miles. What is now her direct distance from the place of her destination, and upon what single course must she sail in order to reach it?

Ans. Dist. 40 miles. Course N. 57° 47′ W.

Ex. 4. A ship from latitude 43° 25′ N., longitude 63° W. sails on the following courses; 1, S. W. by S. 63 miles; 2, S. S. W. ½ W. 45 miles; 3, S. by E. 54 miles; 4, S. W. by W. 74 miles. Required the place of the ship.

Ans. Lat. 40° 19′ N. Long. 65° 23′ 48″ W.

Ex. 5. A ship in 17° 12′ N. latitude bound to a port in 18° 40′ N. latitude and 220 miles to the westward, sails on the following courses; 1, N. W. by W. 73 miles; 2, W. N. W. 40 miles; 3, S. S. W. 18 miles. What is now her distance from the place of her destination, and on what single course must she sail in order to reach it?

Ans. Dist. 126 miles. Course N. 67° W.

Ex. 6. A ship at noon on a certain day was in latitude 10° N. and longitude 41° W. She then sailed on the following courses; 1, N. by E. 25 miles; 2, N. N. E. 32 miles; 3, N. W. 28 miles; 4, N. W. by N. 35 miles. Required the distance sailed on a direct course, and the place of the ship.

Ans. Dist. 105.4 miles. Lat. 11° 43′ N. Long. 41° 22′ 30″ W.

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SECTION V.

MERCATOR'S SAILING.

196. The practical problem in Navigation is to determine accurately, by means of a plane triangle, the difference of latitude and longitude, when the course and distance are known.

The problem has been solved, art. 188, for the difference of latitude, but not for the difference of longitude, which, art. 191 is found only by approximation. And the question returns to find the difference of longitude, by aid of the right angled triangle, in a manner that will be accurate for all latitudes and all courses on which a ship may sail.

Returning to fig. 72, and putting l for the latitude of CF, we have $\mathrm{CF} : \mathrm{KL} :: \cos l : \mathrm{R}$,

or $\mathrm{CF} : \mathrm{KL} :: \mathrm{R} : \sec l$;

whence putting d for the departure CF, and D for KL the corresponding difference of longitude, we have, radius being unity, $\mathrm{D} = d \sec. l$

or, *The difference of longitude is equal to the departure multiplied by the secant of the latitude.*

Let us now construct a right angled triangle AHI (fig. 75) in which AI is taken equal to the distance AD (fig. 72), and the angle HAI equal to the course DAS; then, as before, AH will be equal to the sum of the sides EF of the elementary triangles CEF, or the difference of latitude AS, and HI to the sum of the sides CF of these triangles or the whole departure.

Let Abc (fig. 75) be one of these elementary triangles

equal to CEF, and let the side Ab be extended until, l being the latitude of bc, we have $\quad A p = A b \sec l.$
Draw pq parallel to bc; then from the similarity of the triangles Abc, Apq, we shall have $\quad pq = bc \sec l$
that is, pq will be equal to the difference of longitude corresponding to the departure bc.

Suppose next the sides in all the elementary triangles CEF extended in the same manner as those of Abc, and let the side AH of the triangle AHI be extended to H′ so that AH′ will be equal to the sum of the extended sides Ap of the elementary triangles CEF. Then if from H′ we draw HI′ parallel to HI and meeting AI produced in I′, H′I′ will, it is evident, be equal to the sum of the extended sides pq of the elementary triangles CEF, or to the true difference of longitude corresponding to the sum of the sides CF of these triangles, or to the whole departure HI.

197. The extended difference of latitude AH′ may be found by calculation. It is equal, as we have seen, radius being unity, to the sum of the sides Ap of the elementary triangles CEF, or the sum of the parts into which AS, the difference of latitude, is divided multiplied each by the secant of its latitude.

To facilitate finding the extended difference of latitude corresponding to any proper difference of latitude, tables have been prepared, the construction of which may be explained as follows.

Commencing at the equator, in the formula $Ap = ab \sec l$ let $ab = 1'$; then $Ap = \sec 1'$, that is, the extended difference of latitude, or which is the same thing, the extended meridian corresponding to the first minute of latitude is the natural secant of one minute; that corresponding to the second minute is the natural secant of two minutes, and so on for each minute on the meridian from the equator to the poles.

The extended meridian corresponding, therefore, to 1, 2, 3, &c. minutes of latitude will be

For 1 minute $\sec 1' = 1.0000000$
2 minutes $\sec 1' + \sec 2' = 2.0000002$
3 minutes $\sec 1' + \sec 2' + \sec 3' = 3.0000006$
4 minutes $\sec 1' + \sec 2' + \sec 3' + \sec 4' = 4.0000013$

The values of the extended meridian corresponding to 1, 2, 3, 4, &c. minutes are called *meridional* parts. They are easily found as above, and are usually set down in tables for every minute and degree of the quadrant from 0 to about 80°.

To find from the tables the meridional difference of latitude corresponding to any proper difference of latitude, if the extreme latitudes are both north or both south we subtract the meridional parts corresponding to the less latitude from those corresponding to the greater; if the extreme latitudes are one north and the other south we add together the meridional parts corresponding to each; the result will be the meridional difference of latitude sought.

198. Having thus seen the manner in which the meridional difference of latitude corresponding to any proper difference of latitude may be found, we will now show the use which may be made of it in accomplishing the object proposed.

In the triangles AHI, AH′I′ (fig. 75), we have

$$AH : HI :: AH' : H'I'$$

or, *The proper difference of latitude is to the departure, as the meridional difference of latitude is to the difference of longitude.*

By means of the meridional difference of latitude we can, it is evident, by proportions in plane triangles, find the difference of longitude, when the course and distance are given. Thus the problem proposed is completely solved.

Returning again to fig. 75 we have, it is evident,

$$R : \text{tang } HAI :: AH' : H'I'$$

or, *Radius is to the tangent of the course as the meridional difference of latitude is to the difference of longitude.*

199. These are the fundamental proportions in Mercator's sailing. From them others may be easily derived, adapted to the different cases which may occur. We subjoin some examples.

Ex. 1. A ship in latitude 47° 23′ N. and longitude 10° 17′ W. sails 126 miles on a course S. W. by W. What is the longitude at which she has arrived ?

This example has already been solved case I. art. 193. The difference of latitude there found, 1° 10′ subtracted from 47° 23′ gives 46° 13′ N. for the latitude in.

Meridional parts 47° 23′	= 3236.6
" " 46° 13′	= 3134.3
Merid. diff. lat.	= 102.3

R	10.000000
: tang. 56° 15′	10.175107
: : Merid. diff. lat. 102.3	2.009876
: Diff. long. 153.1 = 2° 33′ 6″	2.184983

Ans. Long. in 12° 50′ 6″.

Ex. 2. A ship in latitude 33° 20′ N., and longitude 27° 10′ W. sails southeasterly, until by observation her latitude is found to be 25° 30′ N., having made 231 miles departure. What is the longitude in and the course on which she has sailed ?

Merid. parts 33° 20′ = 2123.4 ; 25° 30′ = 1583.2

Diff. lat. 470	2.672098
: Depart. 231	2.363612
: : Merid. diff. lat. 540.2	2.732555
: Diff. long. 265.5	2.424069

Merid. diff. lat.	2.732555
: Diff. long.	2.424069
: : R	10.000000
: Tang. Course 26° 10′ 25″	9.691514

Ans. Long. in 22° 44′ 30″ W. Course S. 26° 10′ 25″ E.

Ex. 3. A ship in sight of the Peak of Teneriffe bearing N. N. E. finds her latitude by observation 27° 5′ N. Required the longitude in.

Teneriffe latitude 28° 25′ N. Long. 16° 25′ W.

Merid. parts 28° 25′ = 1779.5
" " 27° 5′ = 1689.1

Ans. 17° 2′ 30″ W.

Ex. 4. A ship, after taking her departure from Cape Clear sails S. S. W., and finds by observation that her longitude is 10° 54′ W. What distance has she sailed, and what is her present latitude ?

Cape Clear lat. 51° 12′ N. Long. 9° 45′ W.

Tang. 22° 30′ : R : : Diff. long. 69 : Merid. diff. lat. = 166.6

Meridional parts 51° 12′ =	3587.9
Subtract Merid. diff. lat.	166.6
Latitude in, Merid parts	3421.3

Seeking in the table of meridional parts for the degrees and parts of a degree corresponding to 3421.3, we find 49° 25′ 31″. Ans. Latitude in 49° 25′ 31″. Dist. 115.2 miles.

Ex. 5. It is required to find the bearing and distance from Land's End to Bermudas, the latitude of the former being 50° 06′ N. and the longitude 6° W., and the latitude of the latter place 31° 20′ N. and the longitude 64° 48′ W.

Meridional parts 50° 6′ = 3483.8
" " 31° 20′ = 1981.4

Ans. Bearing 66° 55′ W. Dist. 2872 miles.

MISCELLANEOUS EXAMPLES.

1. A ship sailing on a N. W. course, at the rate of 5 knots an hour, observes a headland bearing E, and four

hours afterward the same headland bore S. 78° E. What was its distance from the ship at the time last observed?

Ans. 68 miles.

2. A ship at sea, sailing N. N. W. at the rate of $6\frac{1}{2}$ knots an hour, observed at day-break, that she had just passed near a dangerous reef of rocks. Continuing her course for 8 hours, she then made the land bearing N. E. by E. distant 10 leagues. It is required to find the bearing and distance of the reef of rocks from this land.

Ans. Bearing S. 40° $27\frac{1}{2}'$ W. Distance 64.9 miles.

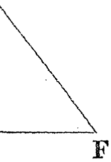
3. Two ships sail from the same port. The first sails N. E. $\frac{1}{2}$ E. 16 miles; the second sails easterly 20 miles, and then finds that the first bears N. N. W. Required the course of the second ship and the distance between the two ships.

Ans. Course S. 72° 27′ E. Dist. 17.5 miles.

4. From a ship under sail, an island is observed to bear N. $22\frac{1}{2}°$ E., and after proceeding N. $67\frac{1}{2}°$ W. 20 miles, the bearing of the same island is found to be N. $56\frac{1}{4}°$ E. The distance from each place of observation is required.

Ans. 29.93, and 36 miles respectively.

5. Coasting along shore, two headlands were observed, the first of which bore N. N. W. and the second N. E. by E.; then steering 20 miles E. N. E. the first headland bore N. W. and the second N. E. What is the distance of the two headlands from each other?



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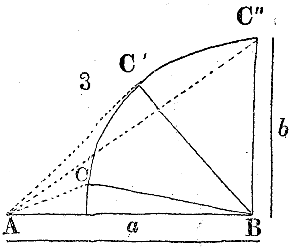

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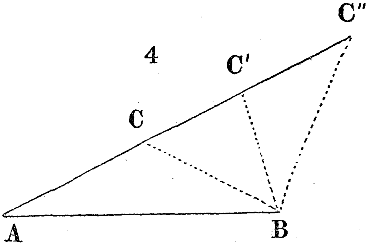

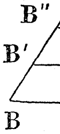

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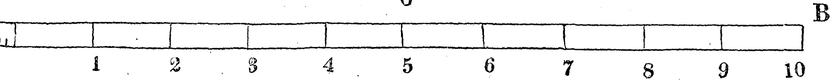

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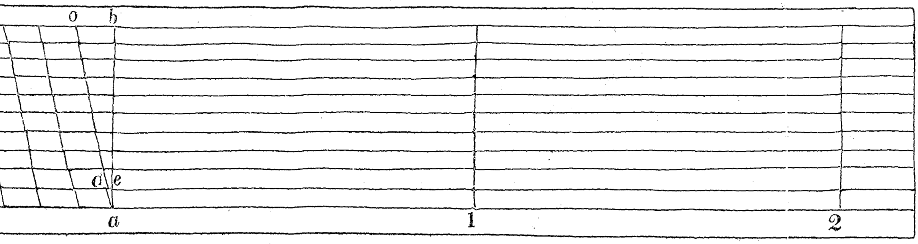

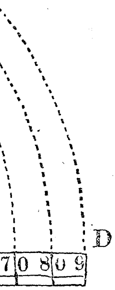

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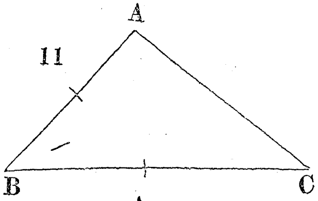

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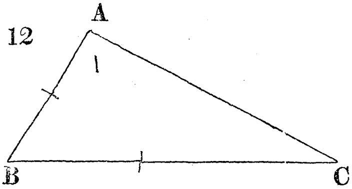

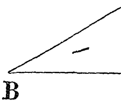

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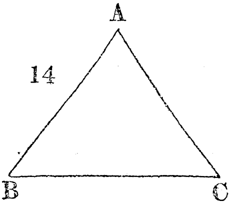

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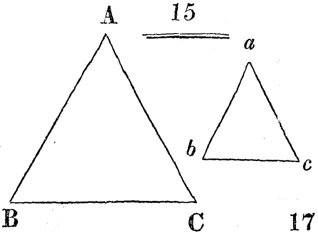

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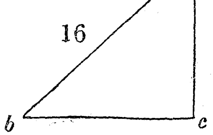

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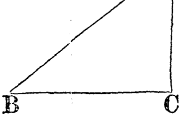

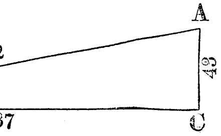

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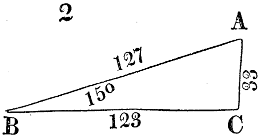

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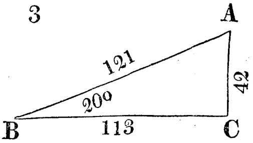

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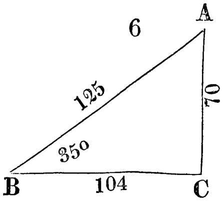

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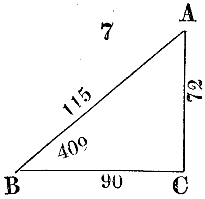

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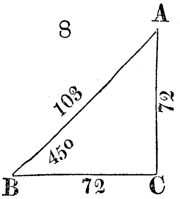

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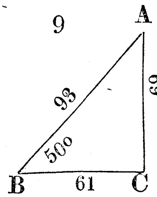

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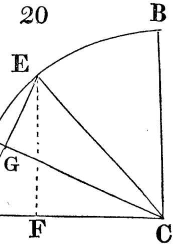

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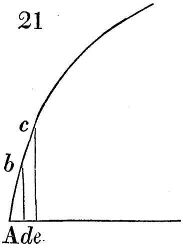

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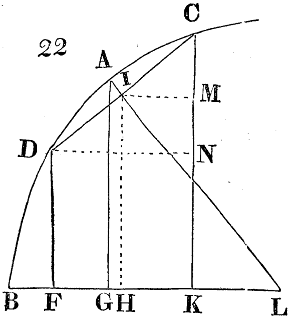

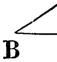

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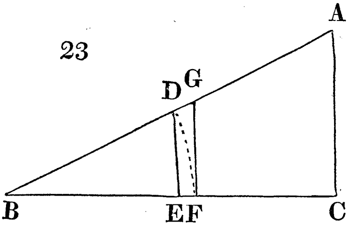

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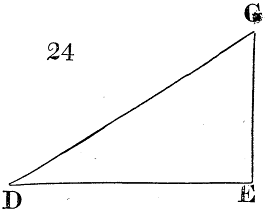

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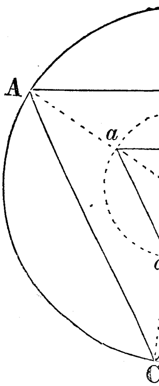

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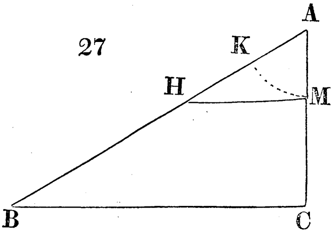

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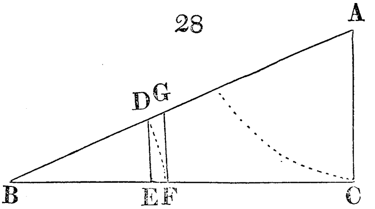

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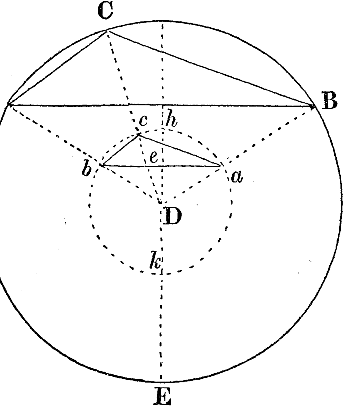

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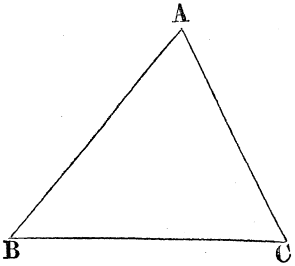

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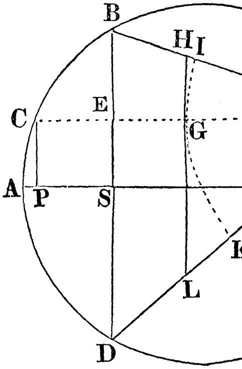

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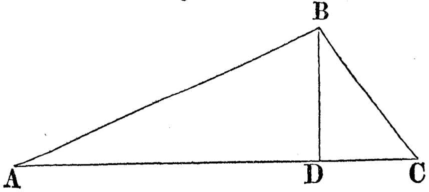

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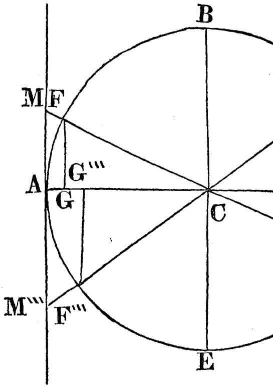

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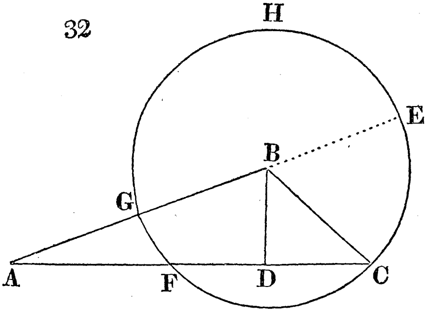

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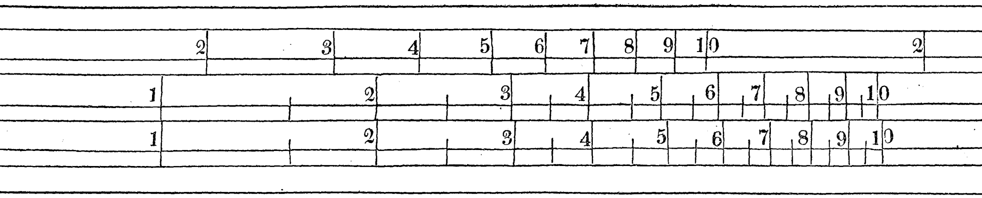

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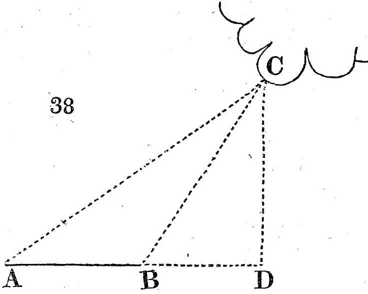

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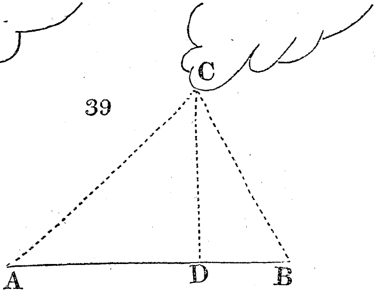

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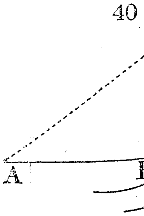

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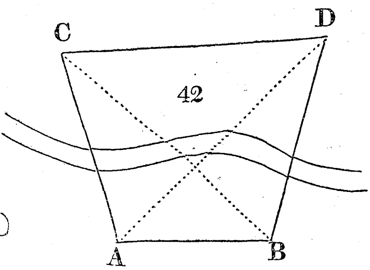

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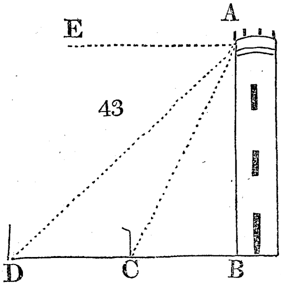

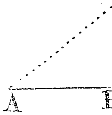

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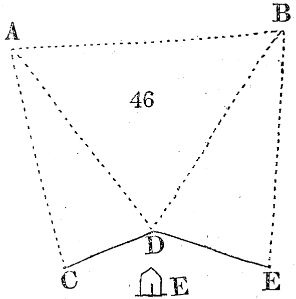

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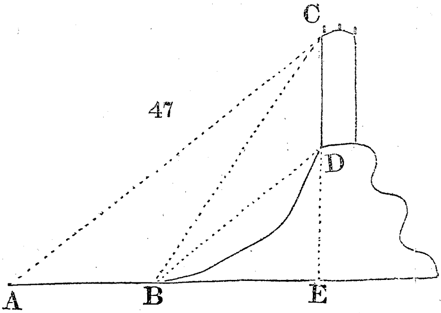

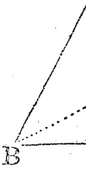

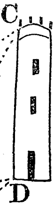

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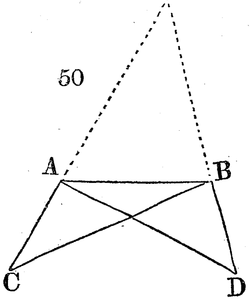

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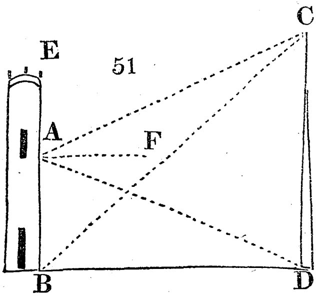

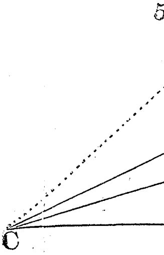

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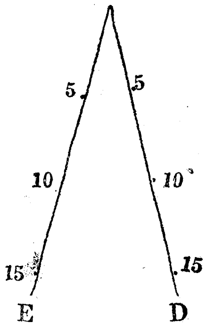
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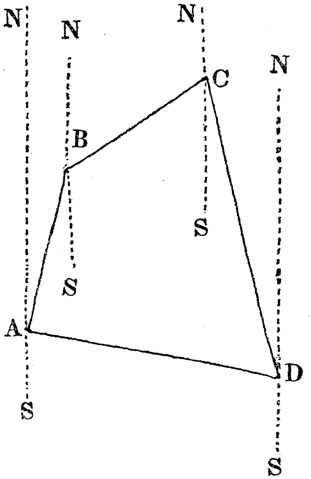
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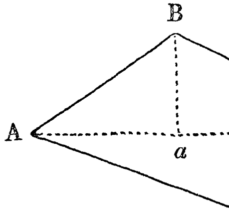
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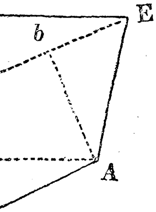
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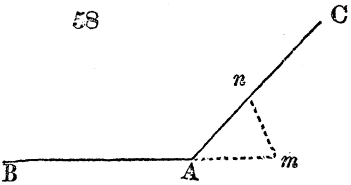
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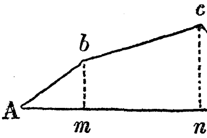
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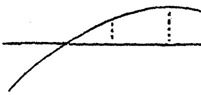

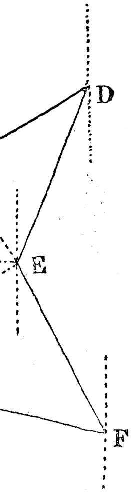
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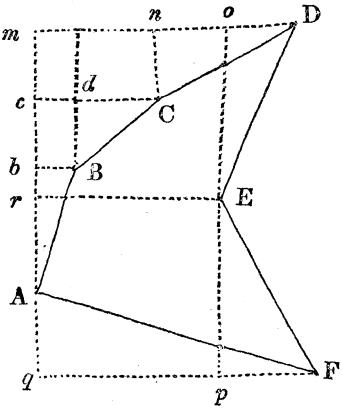
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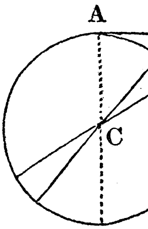
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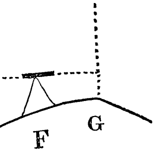
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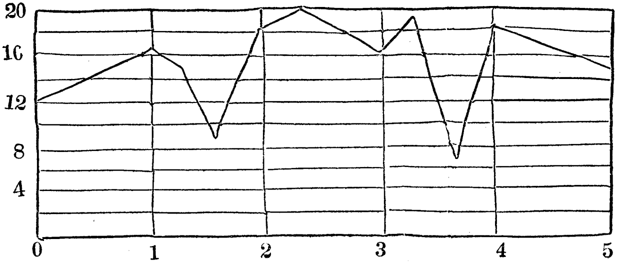
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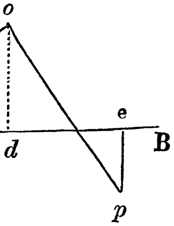
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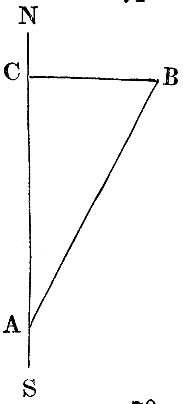
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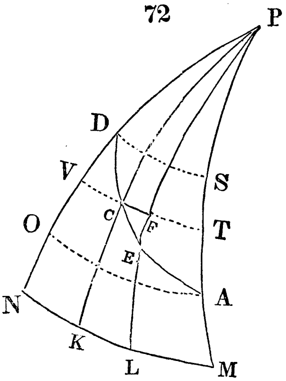
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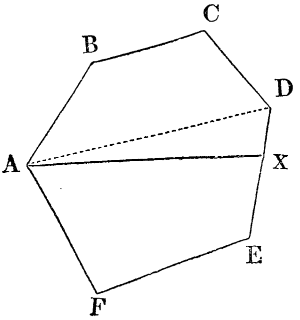
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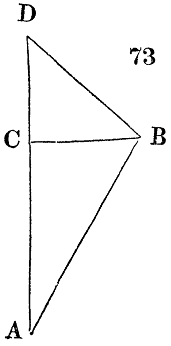
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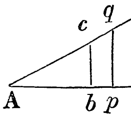
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